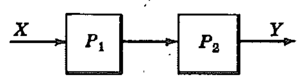
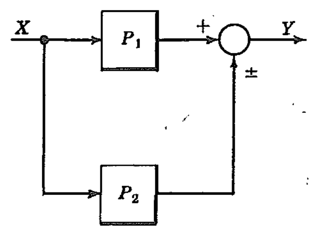
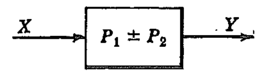
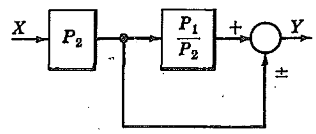
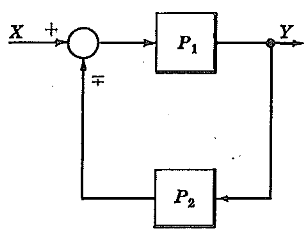
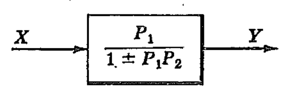
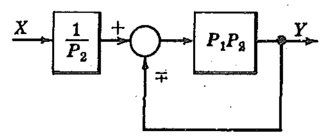
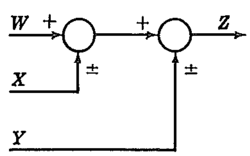
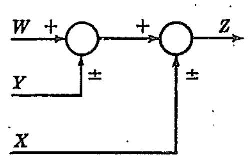
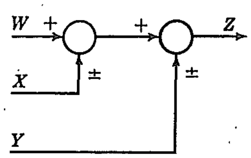
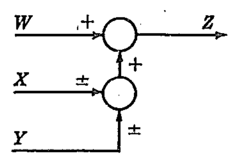
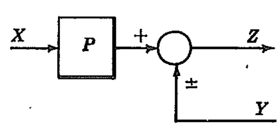
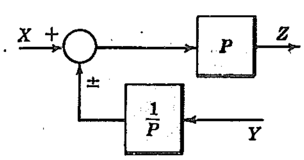
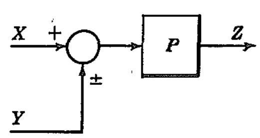
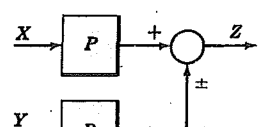
	Transformation	Equation	Block Diagram	Equivalent Block Diagram
1	Combining Blocks in Cascade	$Y = (P_1P_2)X$		
2	Combining Blocks in Parallel; or Eliminating a Forward Loop	$Y = P_1X \pm P_2X$		
3	Removing a Block from a Forward Path	$Y = P_1X \pm P_2X$		
4	Eliminating a Feedback Loop	$Y = P_1(X \mp P_2Y)$		
5	Removing a Block from a Feedback Loop	$Y = P_1(X \mp P_2Y)$		
6a	Rearranging Summing Points	$Z = W \pm X \pm Y$		
6b	Rearranging Summing Points	$Z = W \pm X \pm Y$		
7	Moving a Summing Point Ahead of a Block	$Z = PX \pm Y$		
8	Moving a Summing Point Beyond a Block	$Z = P[X \pm Y]$		

Fig. 7-6

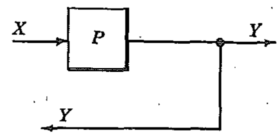
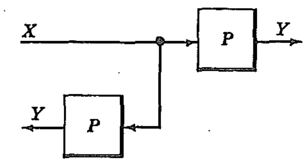
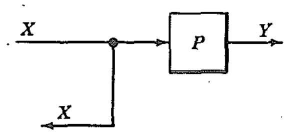
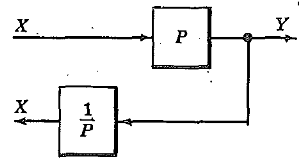
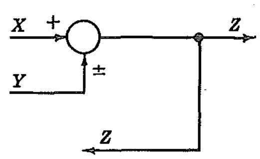
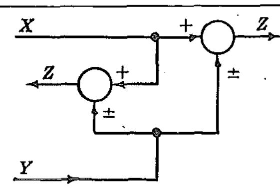
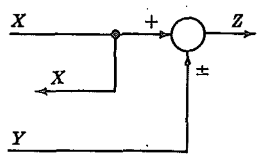
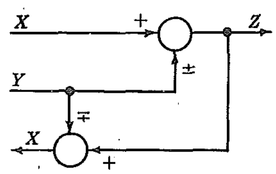
	Transformation	Equation	Block Diagram	Equivalent Block Diagram
9	Moving a Takeoff Point Ahead of a Block	$Y = PX$		
10	Moving a Takeoff Point Beyond a Block	$Y = PX$		
11	Moving a Takeoff Point Ahead of a Summing Point	$Z = X \pm Y$		
12	Moving a Takeoff Point Beyond a Summing Point	$Z = X \pm Y$		

Fig. 7-6 *Continued*

7.6 UNITY FEEDBACK SYSTEMS

Definition 7.7: A **unity feedback system** is one in which the primary feedback b is identically equal to the controlled output c.

EXAMPLE 7.6. $H = 1$ for a linear, unity feedback system (Fig. 7-7).

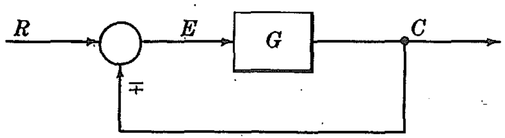


Fig. 7-7

Any feedback system with only linear time-invariant elements can be put into the form of a unity feedback system by using Transformation 5.

EXAMPLE 7.7.

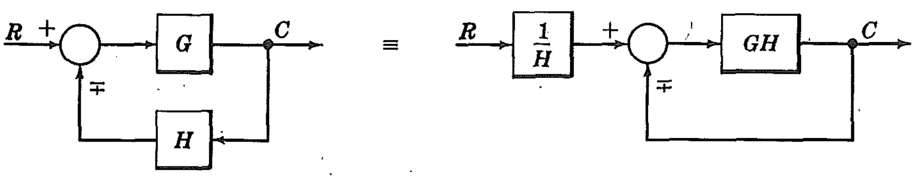


Fig. 7-8

The characteristic equation for the unity feedback system, determined from $1 \pm G = 0$, is

$$D_G \pm N_G = 0 \tag{7.7}$$

where D_G is the denominator and N_G the numerator of G.

7.7 SUPERPOSITION OF MULTIPLE INPUTS

Sometimes it is necessary to evaluate system performance when several inputs are simultaneously applied at different points of the system.

When multiple inputs are present in a *linear* system, each is treated independently of the others. The output due to all stimuli acting together is found in the following manner. We assume zero initial conditions, as we seek the system response only to inputs.

Step 1: Set all inputs except one equal to zero.

Step 2: Transform the block diagram to canonical form, using the transformations of Section 7.5.

Step 3: Calculate the response due to the chosen input acting alone.

Step 4: Repeat Steps 1 to 3 for each of the remaining inputs.

Step 5: Algebraically add all of the responses (outputs) determined in Steps 1 to 4. This sum is the total output of the system with all inputs acting simultaneously.

We reemphasize here that the above superposition process is dependent on the system being linear.

EXAMPLE 7.8. We determine the output C due to inputs U and R for Fig. 7-9.

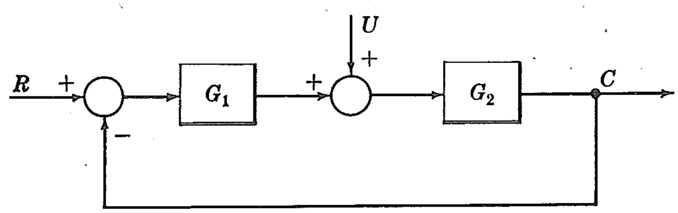
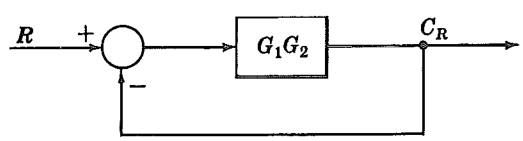
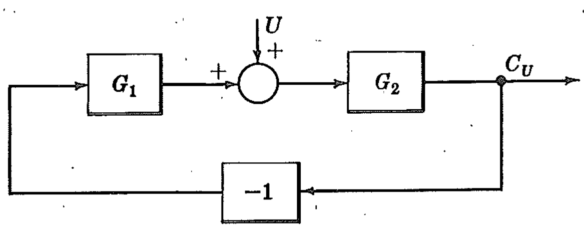


Fig. 7-9

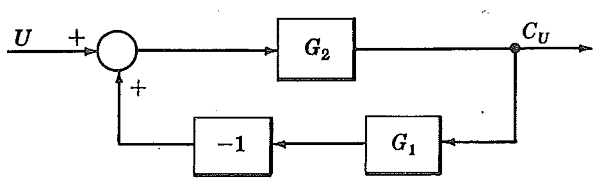
Step 1: Put $U \equiv 0$.
Step 2: The system reduces to



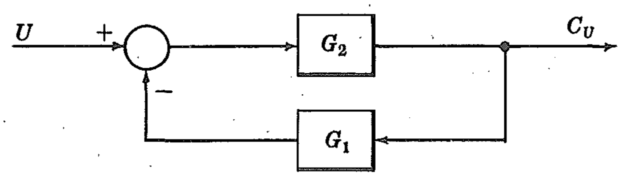
Step 3: By Equation (*7.3*), the output C_R due to input R is $C_R = [G_1G_2/(1 + G_1G_2)]R$.
Step 4a: Put $R = 0$.
Step 4b: Put -1 into a block, representing the negative feedback effect:



Rearrange the block diagram:



Let the -1 block be absorbed into the summing point:



Step 4c: By Equation (7.3), the output C_U due to input U is $C_U = [G_2/(1+G_1G_2)]U$.

Step 5: The total output is

$$C = C_R + C_U = \left[\frac{G_1G_2}{1+G_1G_2}\right]R + \left[\frac{G_2}{1+G_1G_2}\right]U = \left[\frac{G_2}{1+G_1G_2}\right][G_1R+U]$$

7.8 REDUCTION OF COMPLICATED BLOCK DIAGRAMS

The block diagram of a practical feedback control system is often quite complicated. It may include several feedback or feedforward loops, and multiple inputs. By means of systematic block diagram reduction, every multiple loop linear feedback system may be reduced to canonical form. The techniques developed in the preceding paragraphs provide the necessary tools.

The following general steps may be used as a basic approach in the reduction of complicated block diagrams. Each step refers to specific transformations listed in Fig. 7-6.

Step 1: Combine all cascade blocks using Transformation 1.

Step 2: Combine all parallel blocks using Transformation 2.

Step 3: Eliminate all minor feedback loops using Transformation 4.

Step 4: Shift summing points to the left and takeoff points to the right of the major loop, using Transformations 7, 10, and 12.

Step 5: Repeat Steps 1 to 4 until the canonical form has been achieved for a particular input.

Step 6: Repeat Steps 1 to 5 for each input, as required.

Transformations 3, 5, 6, 8, 9, and 11 are sometimes useful, and experience with the reduction technique will determine their application.

EXAMPLE 7.9. Let us reduce the block diagram (Fig. 7-10) to canonical form.

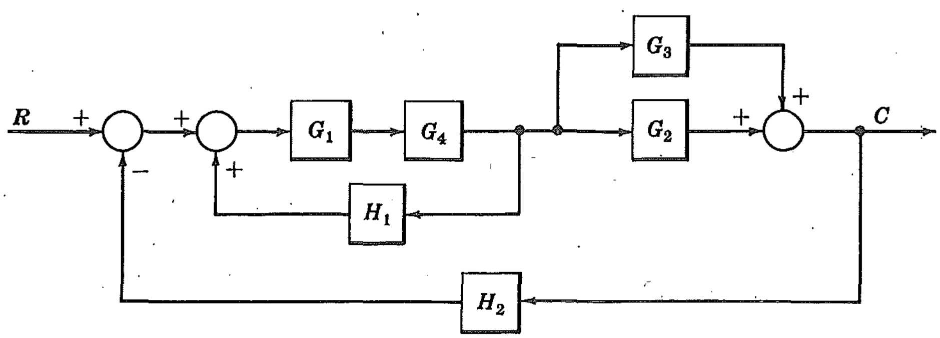
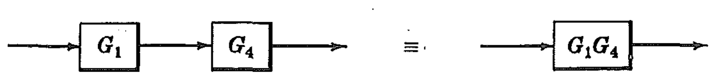
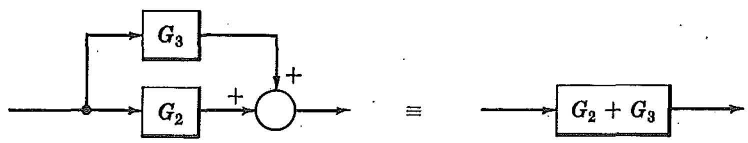


Fig. 7-10

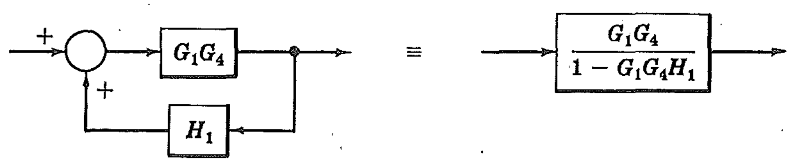
Step 1:



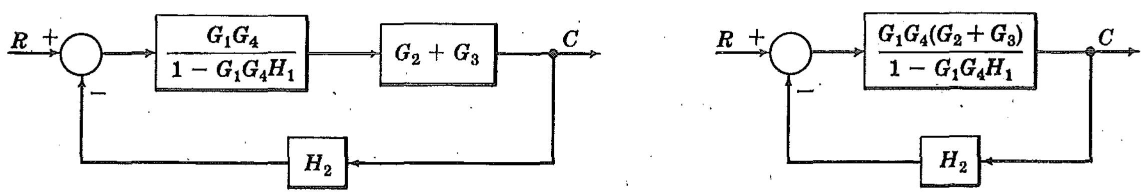
Step 2:



Step 3:



Step 4: Does not apply.
Step 5:



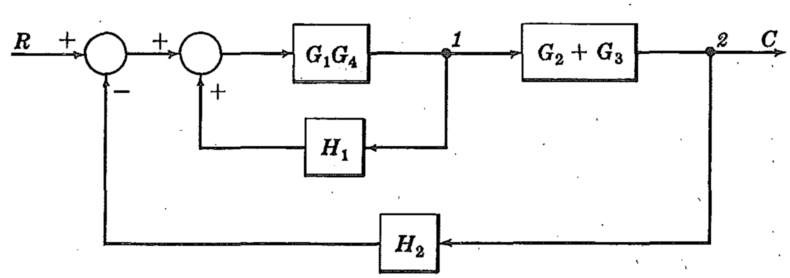
Step 6: Does not apply.

An occasional requirement of block diagram reduction is the isolation of a particular block in a feedback or feedforward loop. This may be desirable to more easily examine the effect of a particular block on the overall system.

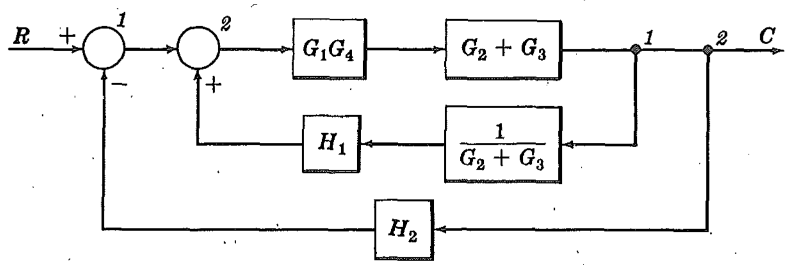
Isolation of a block generally may be accomplished by applying the same reduction steps to the system, but usually in a different order. Also, the block to be isolated cannot be combined with any others.

Rearranging Summing Points (Transformation 6) and Transformations 8, 9, and 11 are especially useful for isolating blocks.

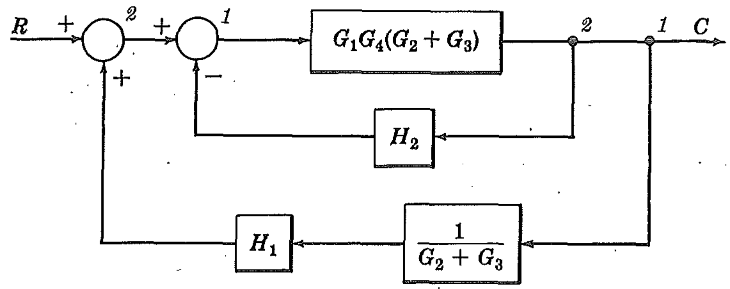
EXAMPLE 7.10. Let us reduce the block diagram of Example 7.9, isolating block H_1.
Steps 1 and 2:



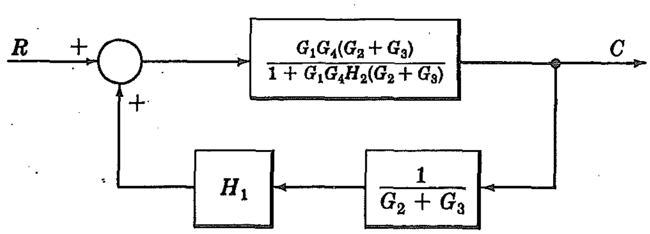
We do not apply Step 3 at this time, but go directly to Step 4, moving takeoff point *1* beyond block $G_2 + G_3$:



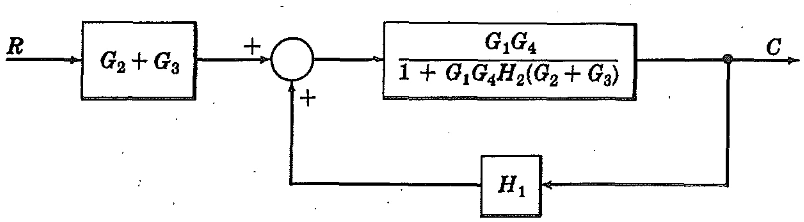
We may now rearrange summing points *1* and *2* and combine the cascade blocks in the forward loop using Transformation 6, then Transformation 1:



Step 3:



Finally, we apply Transformation 5 to remove $1/(G_2 + G_3)$ from the feedback loop:



Note that the same result could have been obtained after applying Step 2 by moving takeoff point *2 ahead* of $G_2 + G_3$, instead of takeoff point *1 beyond* $G_2 + G_3$. Block $G_2 + G_3$ has the same effect on the control ratio C/R whether it directly follows R or directly precedes C.

Solved Problems

BLOCKS IN CASCADE

7.1. Prove Equation (*7.1*) for blocks in cascade.

The block diagram for n transfer functions $G_1, G_2, \ldots, G_n$ in cascade is given in Fig. 7-11.

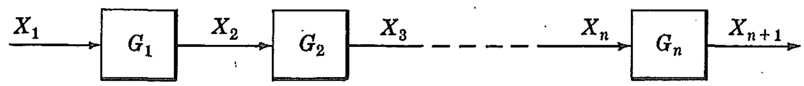


Fig. 7-11

The output transform for any block is equal to the input transform multiplied by the transfer function (see Section 6.1). Therefore $X_2 = X_1G_1$, $X_3 = X_2G_2, \ldots, X_n = X_{n-1}G_{n-1}$, $X_{n+1} = X_nG_n$. Combining these equations, we have

$$X_{n+1} = X_nG_n = X_{n-1}G_{n-1}G_n = \cdots = X_1G_1G_2 \cdots G_{n-1}G_n$$

Dividing both sides by X_1, we obtain $X_{n+1}/X_1 = G_1G_2 \cdots G_{n-1}G_n$.

7.2. Prove the commutativity of blocks in cascade, Equation (*7.2*).

Consider two blocks in cascade (Fig. 7-12):

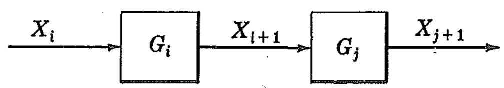


Fig. 7-12

From Equation (*6.1*) we have $X_{i+1} = X_iG_i = G_iX_i$ and $X_{j+1} = X_{i+1}G_j = G_jX_{i+1}$. Therefore $X_{j+1} = (X_iG_i)G_j = X_iG_iG_j$. Dividing both sides by X_i, $X_{j+1}/X_i = G_iG_j$.

Also, $X_{j+1} = G_j(G_iX_i) = G_jG_iX_i$. Dividing again by X_i, $X_{j+1}/X_i = G_jG_i$. Thus $G_iG_j = G_jG_i$.

This result is extended by mathematical induction to any finite number of transfer functions (blocks) in cascade.

7.3. Find X_n/X_1 for each of the systems in Fig. 7-13.

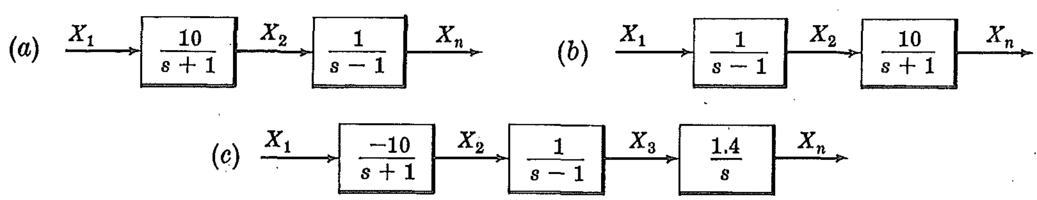


Fig. 7-13

(*a*) One way to work this problem is to first write X_2 in terms of X_1:

$$X_2 = \left(\frac{10}{s+1}\right)X_1$$

Then write X_n in terms of X_2:

$$X_n = \left(\frac{1}{s-1}\right)X_2 = \left(\frac{1}{s-1}\right)\left(\frac{10}{s+1}\right)X_1$$

Multiplying out and dividing both sides by X_1, we have $X_n/X_1 = 10/(s^2-1)$.

A shorter method is as follows. We know from Equation (*7.1*) that two blocks can be reduced to one by simply multiplying their transfer functions. Also, the transfer function of a single block is its output-to-input transform. Hence

$$\frac{X_n}{X_1}=\left(\frac{1}{s-1}\right)\left(\frac{10}{s+1}\right)=\frac{10}{s^2-1}$$

(*b*) This system has the same transfer function determined in part (*a*) because multiplication of transfer functions is commutative.

(*c*) By Equation (*7.1*), we have

$$\frac{X_n}{X_1}=\left(\frac{-10}{s+1}\right)\left(\frac{1}{s-1}\right)\left(\frac{1.4}{s}\right)=\frac{-14}{s(s^2-1)}$$

7.4. The transfer function of Fig. 7-14a is $\omega_0/(s+\omega_0)$, where $\omega_0=1/RC$. Is the transfer function of Fig. 7-14b equal to $\omega_0^2/(s+\omega_0)^2$? Why?

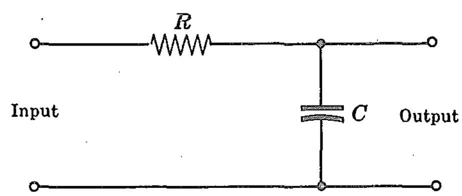


Fig. 7-14a

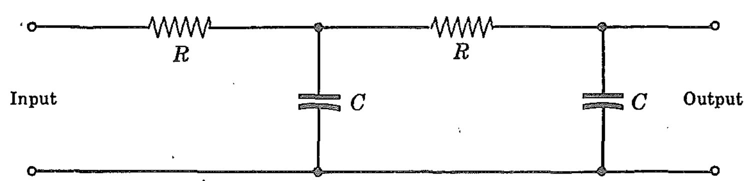


Fig. 7-14b

No. If two networks are connected in series (Fig. 7-15) the second loads the first by drawing current from it. Therefore Equation (*7.1*) cannot be directly applied to the combined system. The correct transfer function for the connected networks is $\omega_0^2/(s^2+3\omega_0 s+\omega_0^2)$ (see Problem 6.16), and this is *not* equal to $(\omega_0/(s+\omega_0))^2$.

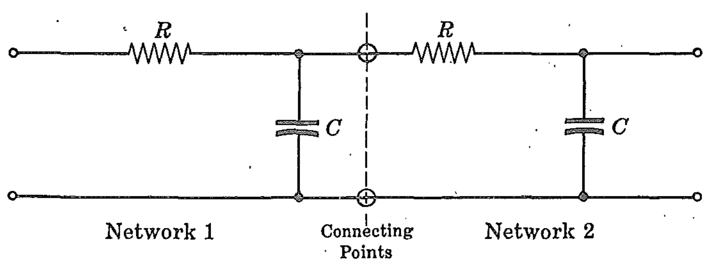


Fig. 7-15

CANONICAL FEEDBACK CONTROL SYSTEMS

7.5. Prove Equation (*7.3*), $C/R=G/(1\pm GH)$.

The equations describing the canonical feedback system are taken directly from Fig. 7-16. They are given by $E=R\mp B$, $B=HC$, and $C=GE$. Substituting one into the other, we have

$$\begin{aligned}C&=G(R\mp B)=G(R\mp HC)\\&=GR\mp GHC=GR+(\mp GHC)\end{aligned}$$

Subtracting $(\mp GHC)$ from both sides, we obtain $C\pm GHC=GR$ or $C/R=G/(1\pm GH)$.

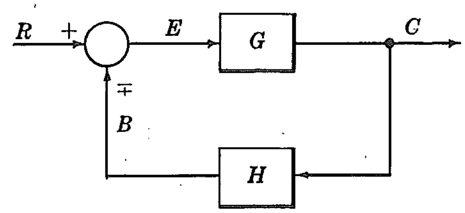


Fig. 7-16

7.6. Prove Equation (*7.4*), $E/R = 1/(1 \pm GH)$.

From the preceding problem, we have $E = R \mp B$, $B = HC$, and $C = GE$. Then $E = R \mp HC = R \mp HGE$, $E \pm GHE = R$, and $E/R = 1/(1 \pm GH)$.

7.7. Prove Equation (*7.5*), $B/R = GH/(1 \pm GH)$.

From $E = R \mp B$, $B = HC$, and $C = GE$, we obtain $B = HGE = HG(R \mp B) = GHR \mp GHB$. Then $B \pm GHB = GHR$, $B = GHR/(1 \pm GH)$, and $B/R = GH/(1 \pm GH)$.

7.8. Prove Equation (*7.6*), $D_{GH} \pm N_{GH} = 0$.

The characteristic equation is usually obtained by setting $1 \pm GH = 0$. (See Problem 7.9 for an exception.) Putting $GH \equiv N_{GH}/D_{GH}$, we obtain $D_{GH} \pm N_{GH} = 0$.

7.9. Determine (*a*) the loop transfer function, (*b*) the control ratio, (*c*) the error ratio, (*d*) the primary feedback ratio, (*e*) the characteristic equation, for the feedback control system in which K_1 and K_2 are constants (Fig. 7-17).

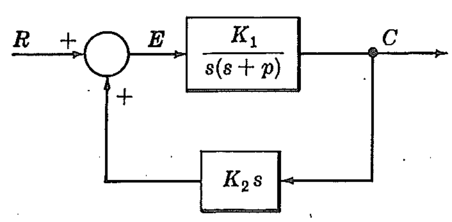


Fig. 7-17

(*a*) The loop transfer function is equal to GH.

Hence $$GH = \left[\frac{K_1}{s(s+p)}\right] K_2 s = \frac{K_1 K_2}{s+p}$$

(*b*) The control ratio, or closed-loop transfer function, is given by Equation (*7.3*) (with a minus sign for positive feedback):

$$\frac{C}{R} = \frac{G}{1 - GH} = \frac{K_1}{s(s+p-K_1K_2)}$$

(*c*) The error ratio, or actuating signal ratio, is given by Equation (*7.4*):

$$\frac{E}{R} = \frac{1}{1-GH} = \frac{1}{1 - K_1K_2/(s+p)} = \frac{s+p}{s+p-K_1K_2}$$

(*d*) The primary feedback ratio is given by Equation (*7.5*):

$$\frac{B}{R} = \frac{GH}{1-GH} = \frac{K_1K_2}{s+p-K_1K_2}$$

(*e*) The characteristic equation is given by the denominator of C/R above, $s(s+p-K_1K_2) = 0$. In this case, $1 - GH = s + p - K_1K_2 = 0$, which is *not* the characteristic equation, because the pole s of G cancels the zero s of H.

BLOCK DIAGRAM TRANSFORMATIONS

7.10. Prove the equivalence of the block diagrams for Transformation 2 (Section 7.5).

The equation in the second column, $Y = P_1 X \pm P_2 X$, governs the construction of the block diagram in the third column, as shown. Rewrite this equation as $Y = (P_1 \pm P_2)X$. The equivalent block diagram in the last column is clearly the representation of this form of the equation (Fig. 7-18)

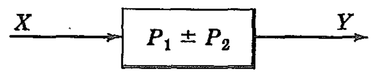


Fig. 7-18

7.11. Repeat Problem 7.10 for Transformation 3.

Rewrite $Y = P_1 X \pm P_2 X$ as $Y = (P_1/P_2)P_2 X \pm P_2 X$. The block diagram for this form of the equation is clearly given in Fig. 7-19.

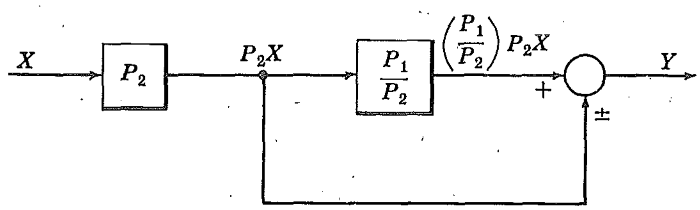


Fig. 7-19

7.12. Repeat Problem 7.10 for Transformation 5.

We have $Y = P_1[X \mp P_2 Y] = P_1 P_2[(1/P_2)X \mp Y]$. The block diagram for the latter form is given in Fig. 7-20.

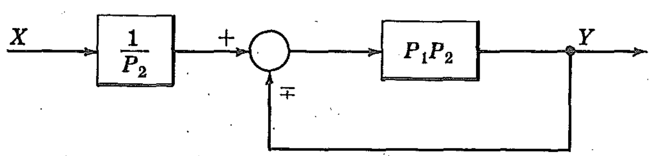


Fig. 7-20

7.13. Repeat Problem 7.10 for Transformation 7.

We have $Z = PX \pm Y = P[X \pm (1/P)Y]$, which yields the block diagram given in Fig. 7-21.

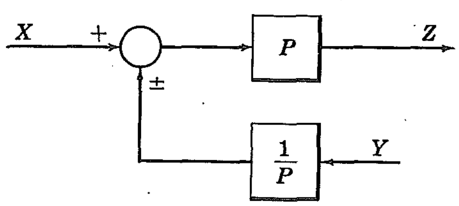


Fig. 7-21

7.14. Repeat Problem 7.10 for Transformation 8.

We have $Z = P(X \pm Y) = PX \pm PY$, whose block diagram is clearly given in Fig. 7-22.

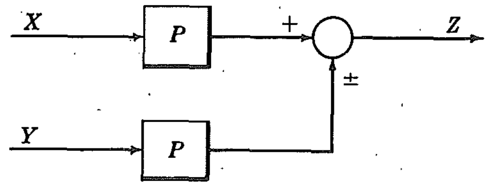


Fig. 7-22

UNITY FEEDBACK SYSTEMS

7.15. Reduce the block diagram given in Fig. 7-23 to unity feedback form and find the system characteristic equation.

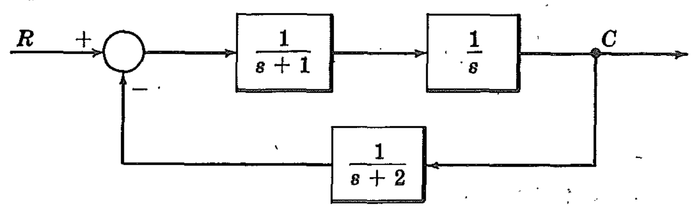


Fig. 7-23

Combining the blocks in the forward path, we obtain Fig. 7-24.

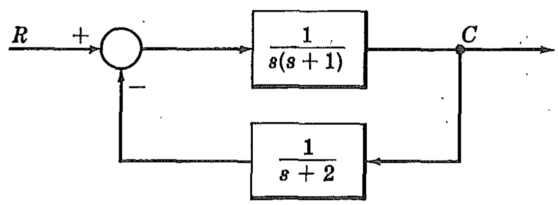


Fig. 7-24

Applying Transformation 5, we have Fig. 7-25.

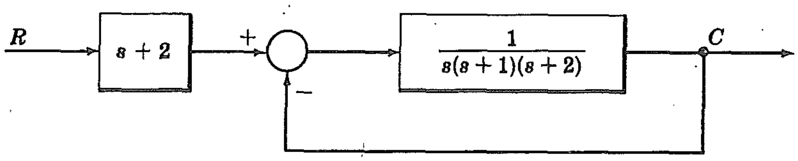


Fig. 7-25

By Equation (7.7), the characteristic equation for this system is $s(s+1)(s+2)+1=0$ or $s^3+3s^2+2s+1=0$.

MULTIPLE INPUTS AND OUTPUTS

7.16. Determine the output C due to U_1, U_2, and R for Fig. 7-26.

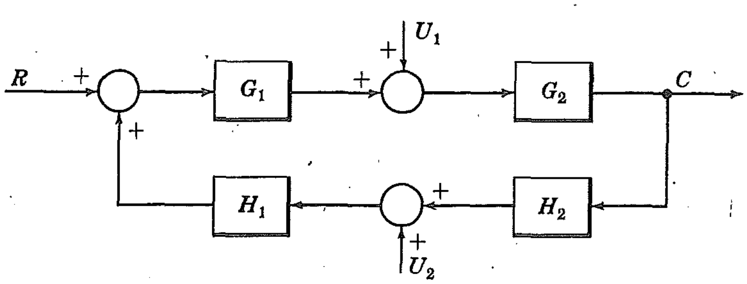


Fig. 7-26

Let $U_1 = U_2 = 0$. After combining the cascaded blocks, we obtain Fig. 7-27, where C_R is the output due to R acting alone. Applying Equation (*7.3*) to this system, $C_R = [G_1G_2/(1 - G_1G_2H_1H_2)]R$.

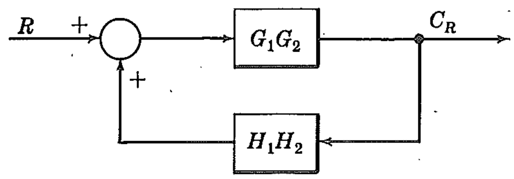


Fig. 7-27

Now let $R = U_2 = 0$. The block diagram is now given in Fig. 7-28, where C_1 is the response due to U_1 acting alone. Rearranging the blocks, we have Fig. 7-29. From Equation (*7.3*), we get $C_1 = [G_2/(1 - G_1G_2H_1H_2)]U_1$.

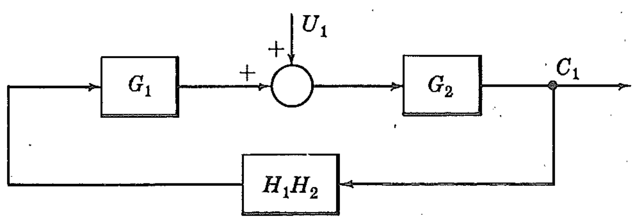


Fig. 7-28

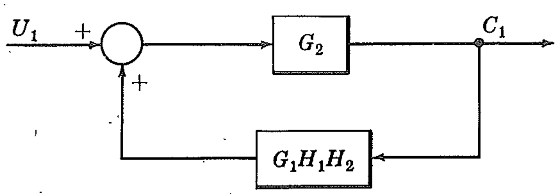


Fig. 7-29

Finally, let $R = U_1 = 0$. The block diagram is given in Fig. 7-30, where C_2 is the response due to U_2 acting alone. Rearranging the blocks, we get Fig. 7-31. Hence $C_2 = [G_1G_2H_1/(1 - G_1G_2H_1H_2)]U_2$.

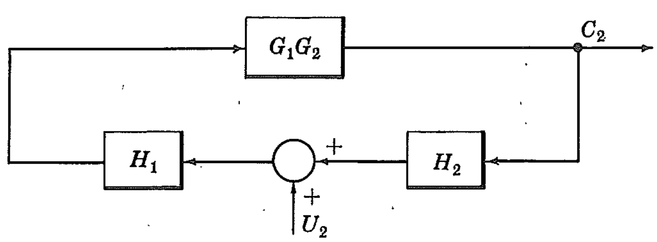


Fig. 7-30

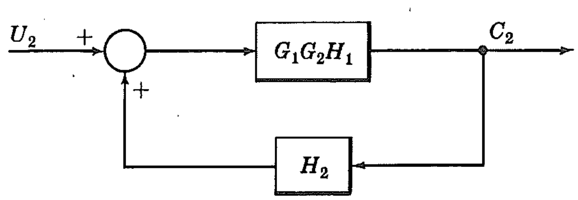


Fig. 7-31

By superposition, the total output is

$$C = C_R + C_1 + C_2 = \frac{G_1G_2R + G_2U_1 + G_1G_2H_1U_2}{1 - G_1G_2H_1H_2}$$

7.17. Figure 7-32 is an example of a multiinput-multioutput system. Determine C_1 and C_2 due to R_1 and R_2.

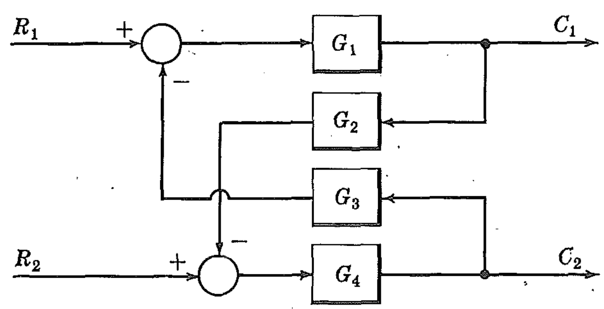


Fig. 7-32

First put the block diagram in the form of Fig. 7-33, ignoring the output C_2.

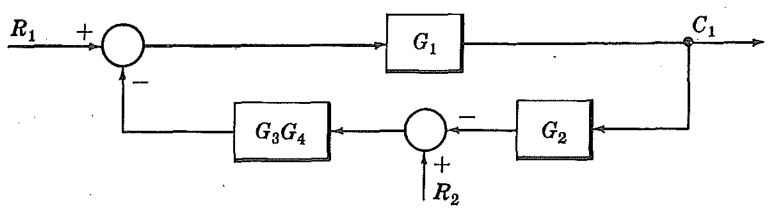


Fig. 7-33

Letting $R_2 = 0$ and combining the summing points, we get Fig. 7-34.

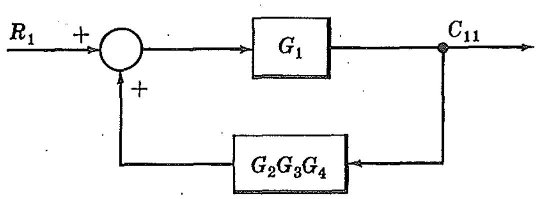


Fig. 7-34

Hence C_{11}, the output at C_1 due to R_1 alone, is $C_{11} = G_1R_1/(1 - G_1G_2G_3G_4)$. For $R_1 = 0$, we have Fig. 7-35.

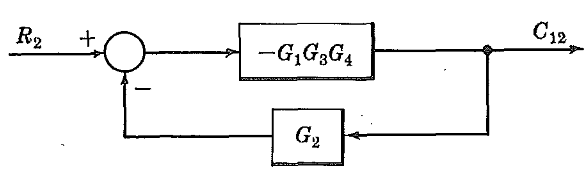


Fig. 7-35

Hence $C_{12} = -G_1G_3G_4R_2/(1 - G_1G_2G_3G_4)$ is the output at C_1 due to R_2 alone. Thus $C_1 = C_{11} + C_{12} = (G_1R_1 - G_1G_3G_4R_2)/(1 - G_1G_2G_3G_4)$.

Now we reduce the original block diagram, ignoring output C_1. First we obtain Fig. 7-36.

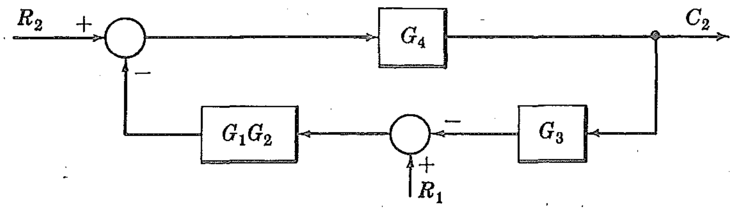


Fig. 7-36

Then we obtain the block diagram given in Fig. 7-37. Hence $C_{22} = G_4R_2/(1 - G_1G_2G_3G_4)$. Next, letting $R_2 = 0$, we obtain Fig. 7-38. Hence $C_{21} = -G_1G_2G_4R_1/(1 - G_1G_2G_3G_4)$. Finally, $C_2 = C_{22} + C_{21} = (G_4R_2 - G_1G_2G_4R_1)/(1 - G_1G_2G_3G_4)$.

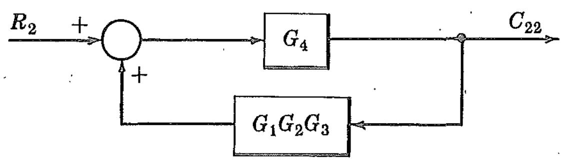


Fig. 7-37

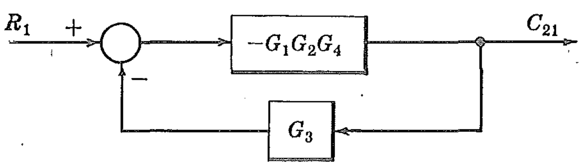


Fig. 7-38

BLOCK DIAGRAM REDUCTION

7.18. Reduce the block diagram given in Fig. 7-39 to canonical form, and find the output transform C. K is a constant.

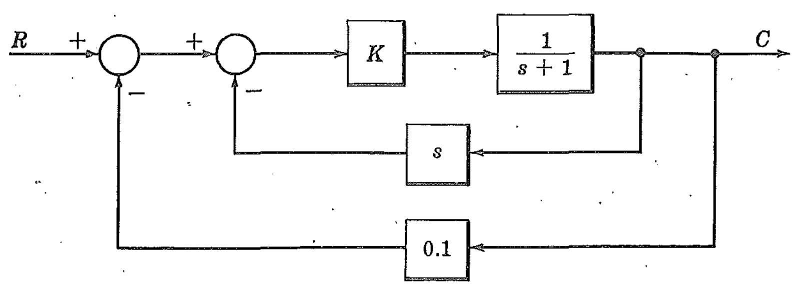


Fig. 7-39

First we combine the cascade blocks of the forward path and apply Transformation 4 to the innermost feedback loop to obtain Fig. 7-40.

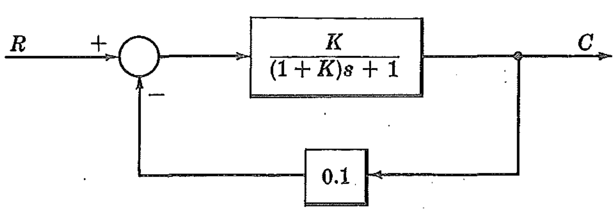


Fig. 7-40

Equation (*7.3*) or the reapplication of Transformation 4 yields $C = KR/[(1+K)s + (1 + 0.1K)]$.

7.19. Reduce the block diagram of Fig. 7-39 to canonical form, isolating block K in the forward loop.

Mathcad

By Transformation 9 we can move the takeoff point ahead of the $1/(s+1)$ block (Fig. 7-41):

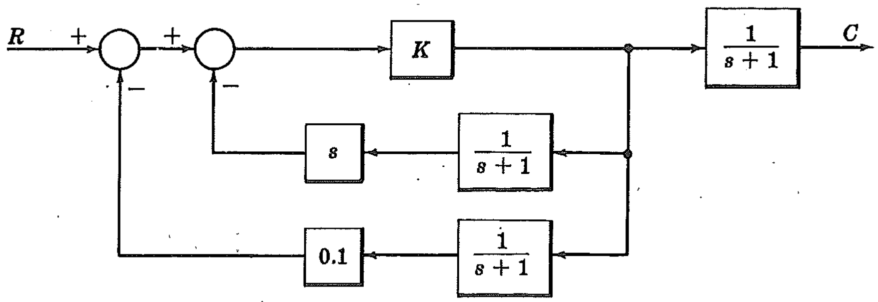


Fig. 7-41

Applying Transformations 1 and 6*b*, we get Fig. 7-42.

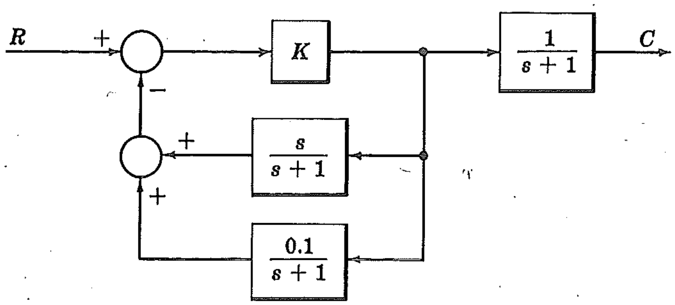


Fig. 7-42

Now we can apply Transformation 2 to the feedback loops, resulting in the final form given in Fig. 7-43.

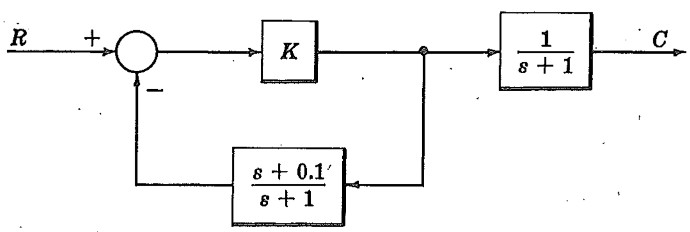


Fig. 7-43

7.20. Reduce the block diagram given in Fig. 7-44 to open-loop form.

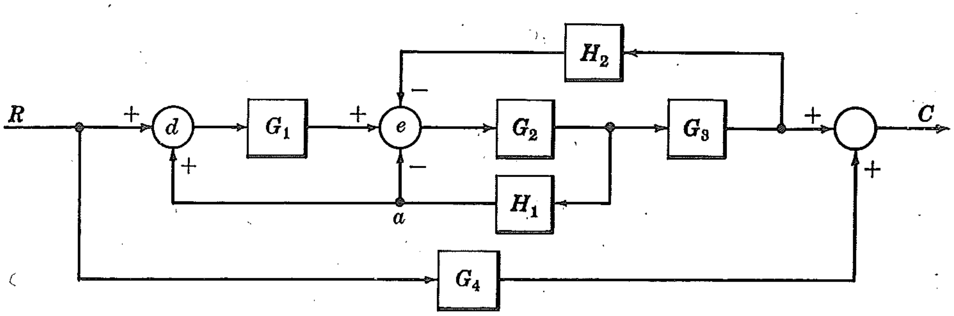


Fig. 7-44

First, moving the leftmost summing point beyond G_1 (Transformation 8), we obtain Fig. 7-45.

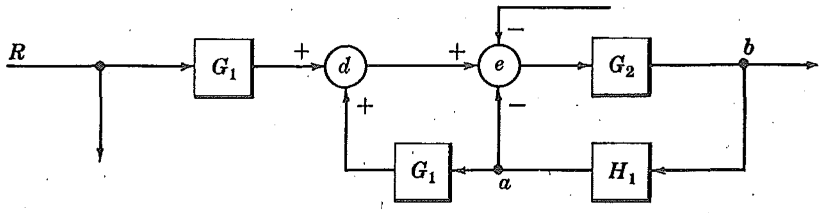


Fig. 7-45

Next, moving takeoff point a beyond G_1, we get Fig. 7-46.

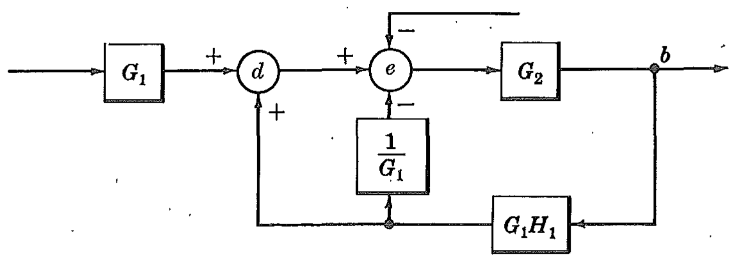


Fig. 7-46

Now, using Transformation 6b, and then Transformation 2, to combine the two lower feedback loops (from G_1H_1) entering d and e, we obtain Fig. 7-47.

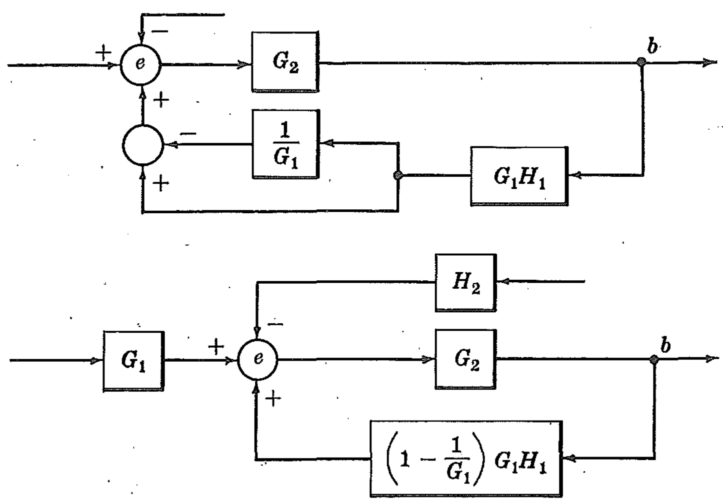
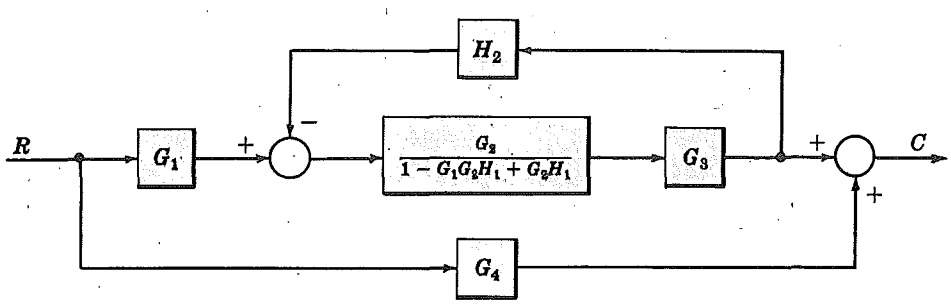
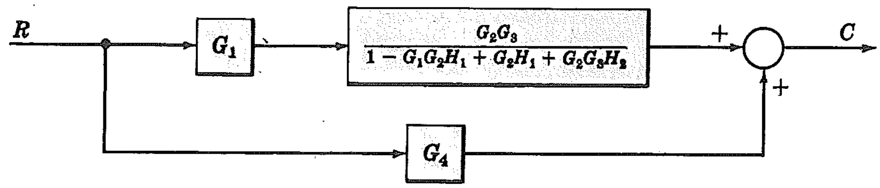


Fig. 7-47

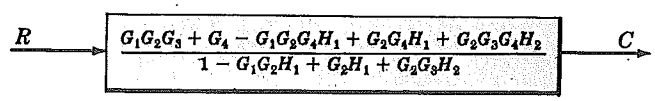
Applying Transformation 4 to this inner loop, the system becomes



Again, applying Transformation 4 to the remaining feedback loop yields



Finally, Transformation 1 and 2 give the open-loop block diagram:



MISCELLANEOUS PROBLEMS

7.21. Show that simple block diagram Transformation 1 of Section 7.5 (combining blocks in cascade) is not valid if the first block is (or includes) a *sampler*.

The output transform $U^*(s)$ of an ideal sampler was determined in Problem 4.39 as

$$U^*(s) = \sum_{k=0}^{\infty} e^{-skT}u(kT)$$

Taking $U^*(s)$ as the input of block P_2 of Transformation 1 of the table, the output transform $Y(s)$ of block P_2 is

$$Y(s) = P_2(s)U^*(s) = P_2(s)\sum_{k=0}^{\infty} e^{-skT}u(kT)$$

Clearly, the input transform $X(s) = U(s)$ cannot be factored from the right-hand side of $Y(s)$, that is, $Y(s) \neq F(s)U(s)$. The same problem occurs if P_1 includes other elements, as well as a sampler.

7.22. Why is the characteristic equation invariant under block diagram transformation?

Block diagram transformations are determined by *rearranging* the input-output equations of one or more of the subsystems that make up the total system. Therefore the final transformed system is governed by the same equations, probably arranged in a different manner than those for the original system.

Now, the characteristic equation is determined from the denominator of the overall system transfer function set equal to zero. Factoring or other rearrangement of the numerator and denominator of the system transfer function clearly does not change it, nor does it alter the denominator set equal to zero.

7.23. Prove that the transfer function represented by C/R in Equation (*7.3*) can be approximated by $\pm 1/H$ when $|G|$ or $|GH|$ are very large.

Dividing the numerator and denominator of $G/(1 \pm GH)$ by G, we get $1\Big/\left(\frac{1}{G} \pm H\right)$. Then

$$\lim_{|G|\to\infty}\left[\frac{C}{R}\right] = \lim_{|G|\to\infty}\left[\frac{1}{\frac{1}{G} \pm H}\right] = \pm\frac{1}{H}$$

Dividing by GH and taking the limit, we obtain

$$\lim_{|GH|\to\infty}\left[\frac{C}{R}\right] = \lim_{|GH|\to\infty}\left[\frac{\frac{1}{H}}{\frac{1}{GH} \pm 1}\right] = \pm\frac{1}{H}$$