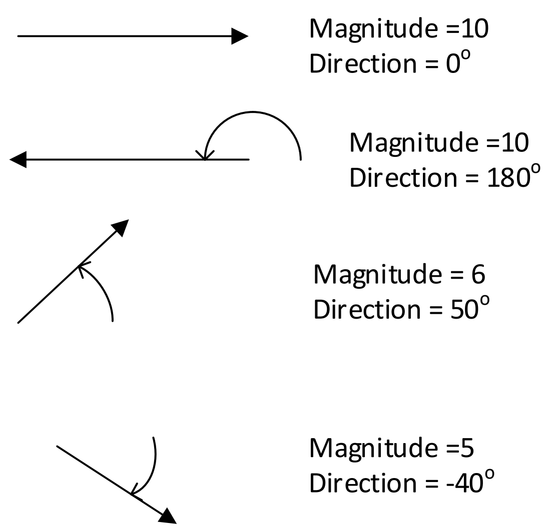
8 Complex Numbers and Sinusoidal Steady State

If I needed to describe the distance between Toledo, Ohio and Cincinnati, I would just answer 200 miles. The direction is straight south and I have driven it a number of times. If you, however, would ask the distance to Indianapolis, Indiana, I would have to ask you whether you were driving down I-75 and then turning right at Dayton or if you were driving to Fort Wayne ad then down I-69. You would need to provide both direction and mileage for the description to Indianapolis. I need more information than just a scalar number of miles. I need direction or some other description indicating direction as well as number of miles.

When we deal with scalar numbers, life is simple. However, when we begin to delve into numbers with a direction component, we begin to look at mathematical concepts with more depth. With a circuit containing a battery, we just add the numbers around a loop and solve an equation. With a sinusoidal waveform, the quantities may be simple or complex, depending on the devices in the circuit. We need to learn to deal with complex numbers when dealing with AC circuits if the circuit contains multiple waveforms that may be out of phase with each other or if inductors or capacitors are involved. Whew, I said it! No longer in Kansas, Dorothy!

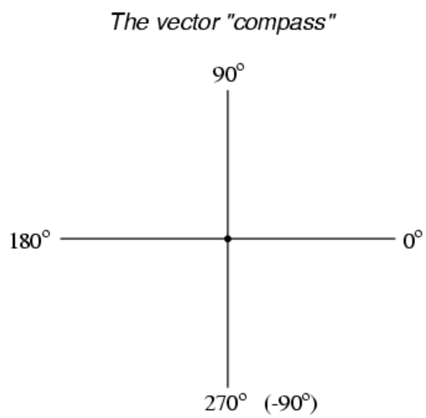
A complex number is a single mathematical quantity that conveys dimensions of amplitude and phase shift.

Complex numbers are easy to grasp graphically. A line with length and angle can represent a complex number. Several examples are shown in the figure below:



A vector has both magnitude and direction.

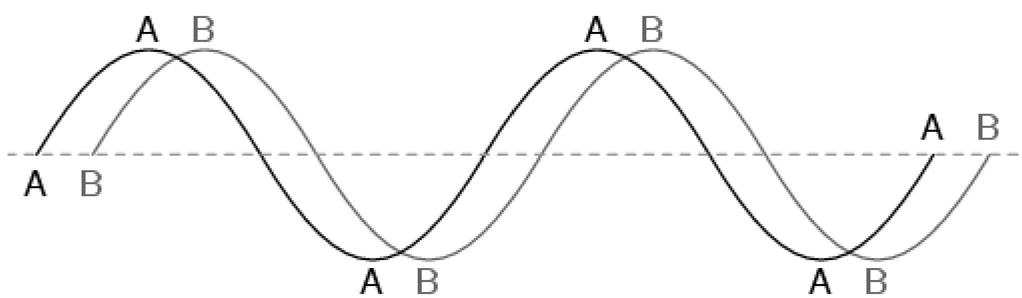
The reference for these vectors is a Cartesian plane similar to the following ‘vector compass’. Magnitude is measured in vector length and angle is measured counterclockwise from the positive direction.



The vector compass

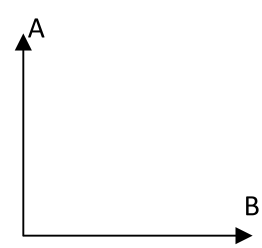
AC Phase

Below is a picture of sine waves that are out of phase with each other. The A wave leads the B wave by a phase angle that is measureable. We can see this on an oscilloscope. The two waves have the same amplitude and frequency but are out of phase with each other.

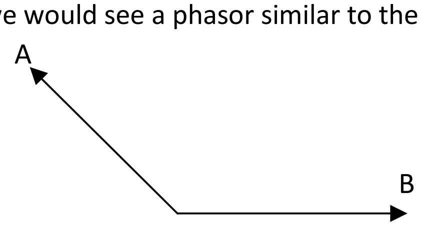


Out of phase waveforms

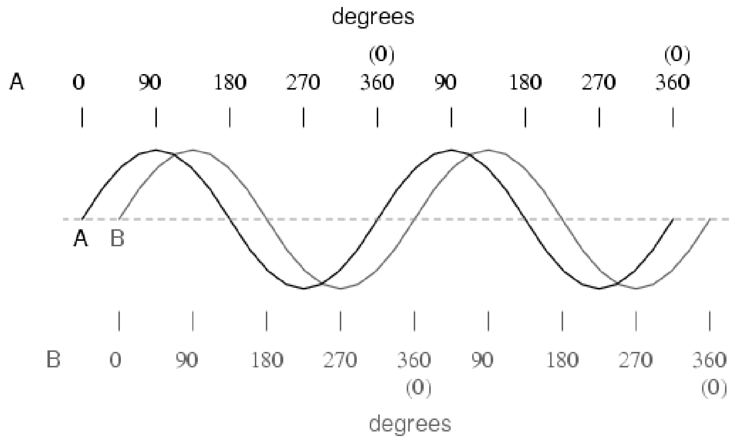
If the phase shift were 90°, we could draw the phasor diagram of the A and B phasors as follows:



If the phase shift were 135°, we would see a phasor similar to the following:

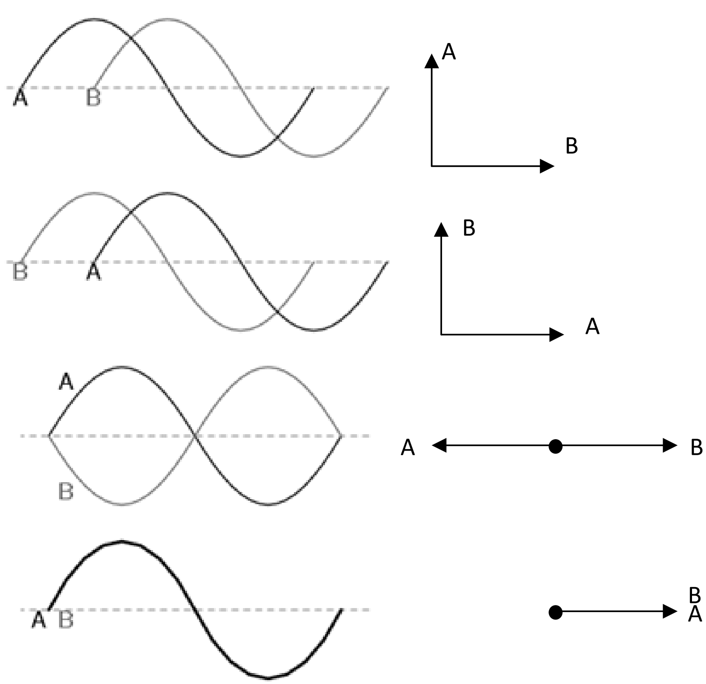


Various voltage phase shifts can be observed in the following figures:



Wave A leads wave B by 45°

Various phase shifts are shown below:



A 90° leading B

B 90° leading A

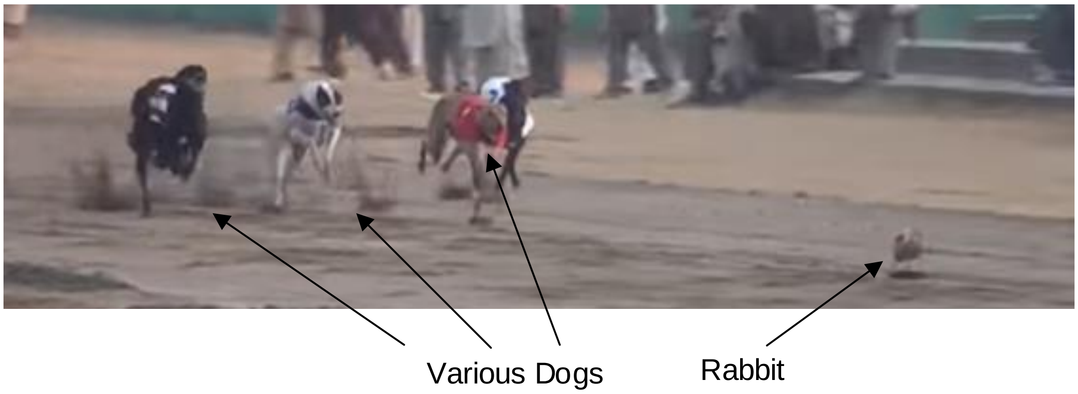
A 180° leading B or B 180° leading A (you pick)

A and B in phase

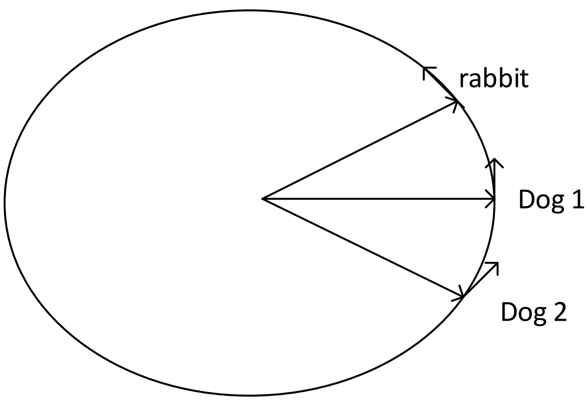
Examples of phase shifts.

Remember that the waveforms are at the same frequency. This is implied in all the phasor diagrams. The reference is where you want to place it. There is no absolute reference.

We veer a little off course with the following examples but they may help you to see that phasors are nothing more than a dog and a rabbit running around a circular track with the rabbit always in the lead by a certain amount.



We could say the dogs and rabbit are out of phase. This would be true since the dogs always 'lag' the rabbit by a certain amount (in degrees).



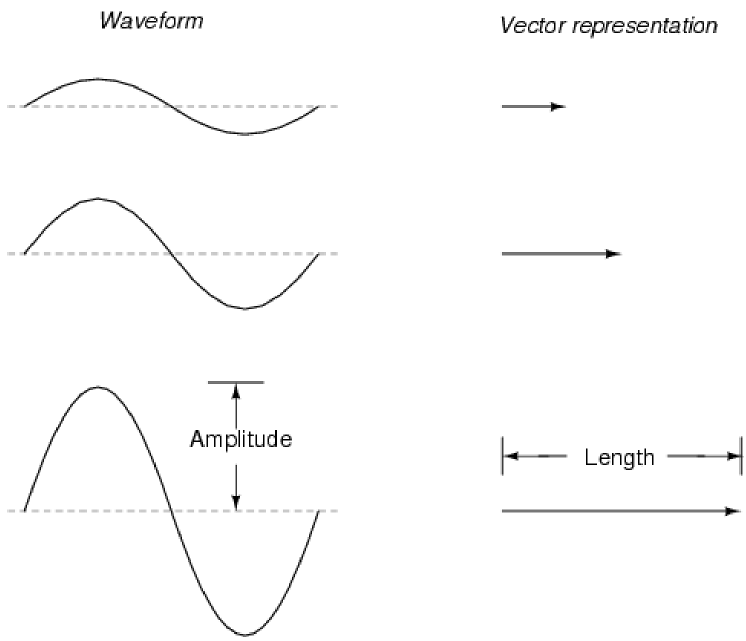
The figure of phasors turning at equal frequencies moving counterclockwise are shown at left. The picture of the dogs chasing the rabbit are the same. They never catch the rabbit and are running in a circle in an order that for now at least is fixed. These phasors (or vectors) have a magnitude and angle.

You could also see phasors as a cat with the hands moving backward (counter-clockwise). The eyes move, the tail moves and the hands move backwards (only for those who see the minute and hour hands as phasors).

While dogs on a track and cats wagging their tail may seem strange, the idea is to get one to think of phasors as both a stationary vector and a moving arrow with length and angular velocity (just like a dog or a cat).

Rectangular Form of a Complex Number

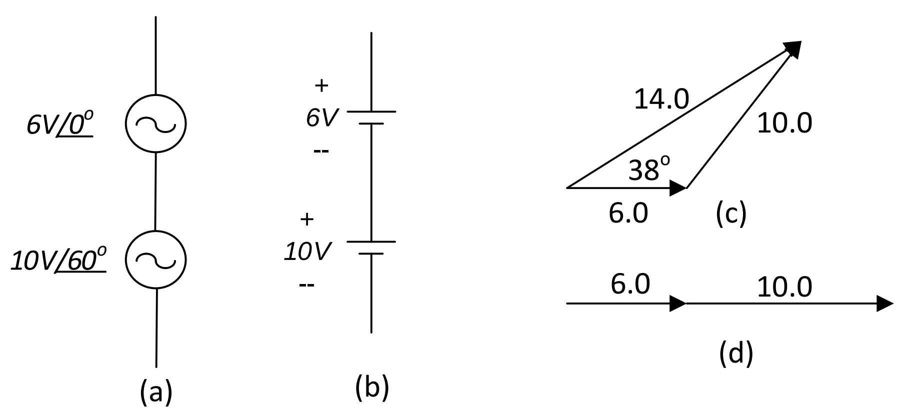
Vectors have amplitude and magnitude. The following waveforms demonstrate the variable nature of sine waves and their vector representation. Notice that each of the waveforms are assumed to be at zero reference point. Amplitude correlates to relative length of the vector.



Length of Vector represents Signal Amplitude

Simple Vector Addition

Remember that vectors are mathematical objects with magnitude and direction. How would you evaluate the sum of the two voltages below in the figure?



While in-phase sinusoidal waves add the same as dc voltage supplies, if the phase is shifted, the numbers add as complex numbers. Calculate voltage across the sum of two vectors, not a linear sum.

This leads us to the subject of Complex Vector Addition

From the example above, to add the two vectors, we simply add the 'x' components of each and the 'y' components of each and find the resultant vector. For the $6\angle 0^{o}$ vector, there is only an 'x' component of 6. For the $10\angle 60^{o}$ vector, the 'x' component is 5.0 and the 'y' component is 8.66. Adding, we get the total 'x' as 6 + 5 = 11 and for the 'y', we get 8.66. Now, find the resultant vector. It is $14\angle 38.21^{o}$.

We use Polar Notation for complex numbers to find the sum of these vectors.

In order to work with complex numbers without drawing vectors, we first need some kind of standard mathematical notation. There are two basic forms of complex number

Polar Form of a Complex Number

We represent sine waves in polar form in order to add them.

We will find that in ac circuits, most of the dc rules still apply, just with complex numbers.

The equation of a sinusoidal wave is:

$$y = M \sin(\vartheta + \varphi)$$

For current, the equation is:

$$i = I_P \sin(\vartheta + \varphi)$$

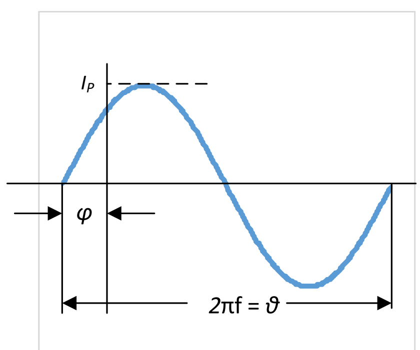
The value at the peak is I_P and the value of phase shift is φ (Greek phi).

For voltage, the equation is:

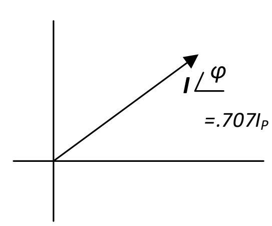
$$v = V_P \sin(\vartheta + \varphi)$$

The value at the peak is V_P and the value of phase shift is φ (Greek phi). The value ϑ represents the time-varying signal and can be seen as $2\pi f t$ where f is the frequency and t is time.

The waveform for the current signal is shown in the figure at left while the phasor form is shown at right. The phasor current is the rms value. The relationship between I_P and I_{RMS} is $\boldsymbol{I_{RMS} = .707\, I_P}$.

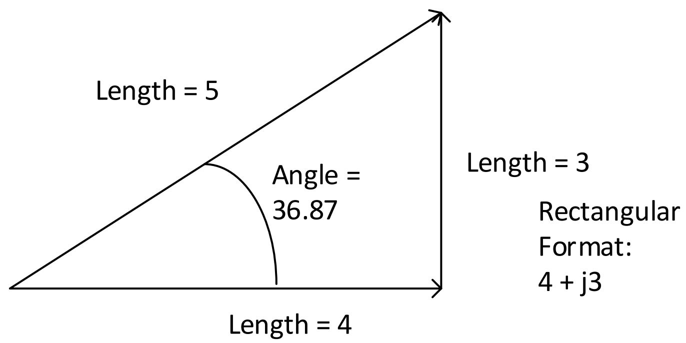


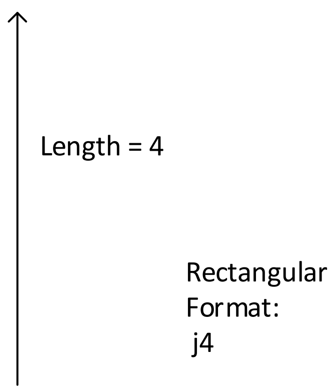
(a)

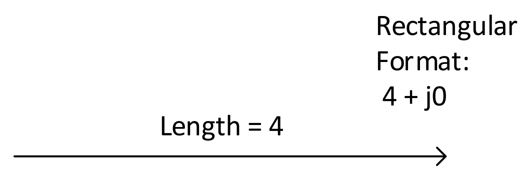


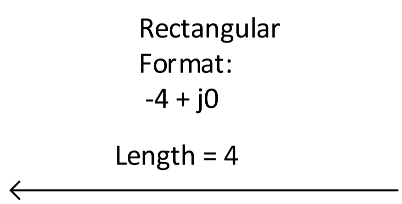
(b)

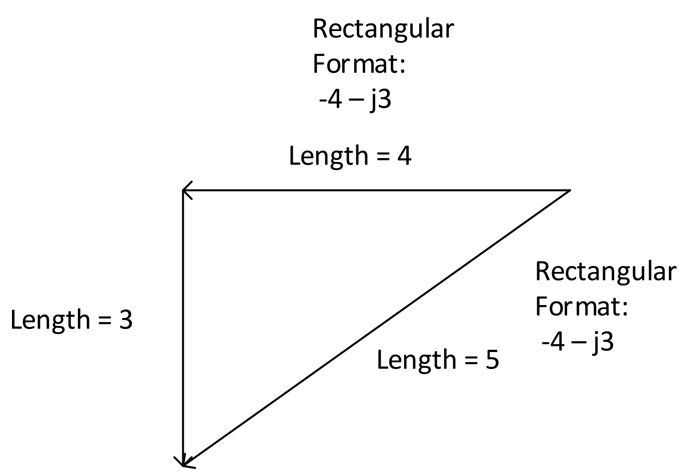
With rectangular form, we can more easily add or subtract vectors. We simply add the ‘x’ components and the ‘y’ components and re-combine the result. In general, we try to add and subtract using rectangular form while we multiply and divide using polar form. Some examples of rectangular form of vectors follow:



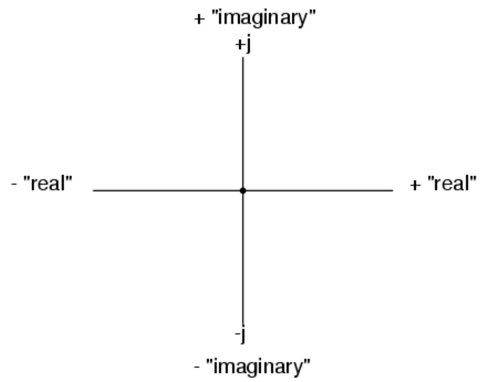








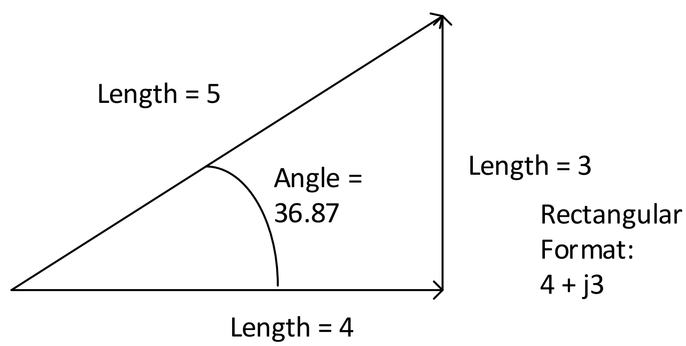
The ‘x’ component is the real component and the ‘y’ component is the imaginary component. The complex plane is shown below with both real and imaginary axes.



Complex Plane

Converting from Polar Form to Rectangular Form

Using a calculator with either R->P or P->R will give the conversion from one form to the other. It is advised for the student to sit down with a calculator and find these functions or a similar method to convert since this will be required for some time in this chapter and the next. Practice…



The above vector may be entered either in rectangular or polar format. It is shown in rectangular format:

4 + j3

To find the polar equivalent, use a calculator to convert:

Enter: 4, R->P, 3, =
Result: 5

Key in x<->y to get:
36.869

This is written: 4 + j3 = 5/<u>36.869</u> (rectangular to polar)

Complex Number Arithmetic

We will be adding complex numbers and it is advised to find a calculator with this capability.

For addition and subtraction of complex numbers, use rectangular form.

$$\begin{array}{r} 2 + j5 \\ + \ \underline{4 - j3} \\ 6 + j2 \end{array}$$

For subtraction, use the same method:

$$\begin{array}{r} 2 + j5 \\ - \ \underline{(4 - j3)} \\ -2 + j8 \end{array}$$

For multiplication and division, polar is preferred. For multiplication, multiply magnitudes and add phase angles. For division, divide the magnitudes and subtract the denominator's phase angle from the numerator's phase angle.

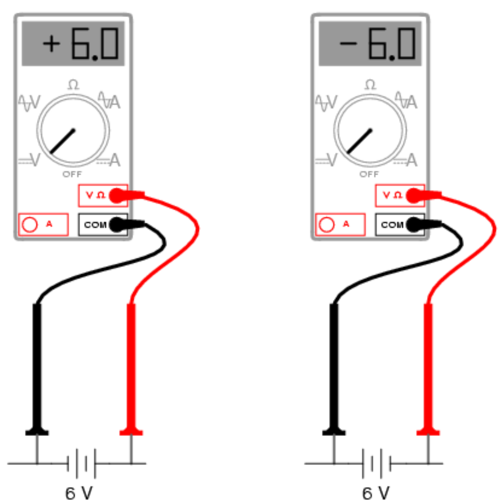
$$(35\underline{/65^o})\ (10\underline{/-10^o}) = 350\underline{/55^o}$$

Division of polar-form complex numbers:

$$\frac{35/65^o}{10/-10^o} = 3.5/75^o$$

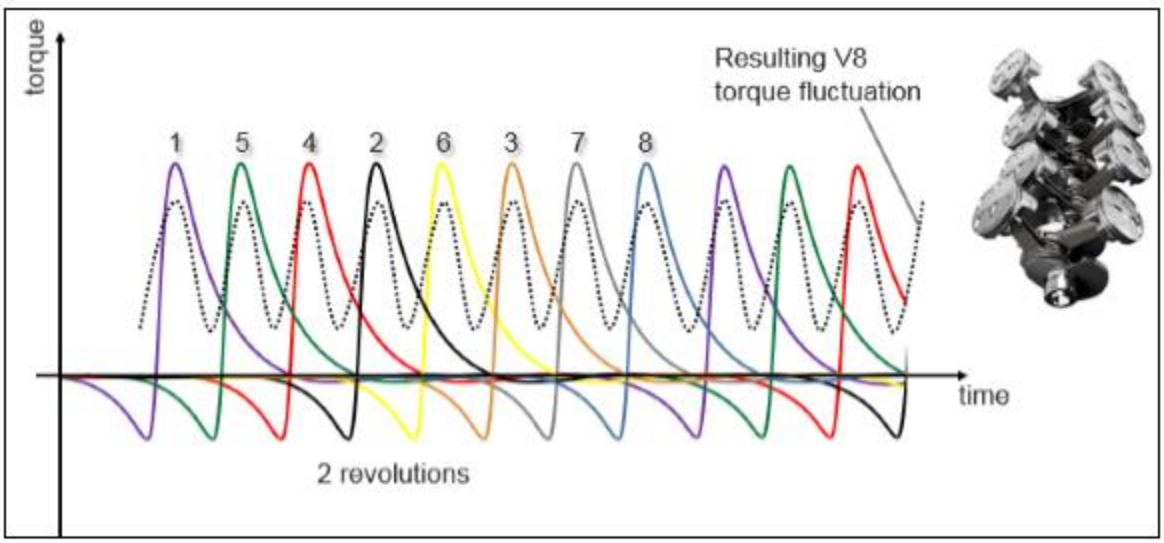
AC "polarity"

If we were to measure the following circuits, we would see the results from the DMM's display. In other words, if we reference the red lead as plus and measure the plus side of the voltage using this lead, we would read +6.0 V. If the leads were reversed, however, the display would show -6.0 V.



Does the same happen when we test an ac circuit? The answer is that it may if the sources are in phase. However, per our example above, a phase shift will make our calculation use complex numbers and the answer will not be a simple addition but rather a vector addition. If we switch the leads from plus to minus, our answer will be 180° phase shifted which would be what we would expect. This change, however, cannot be read with a volt meter or simple DMM but only on an oscilloscope.

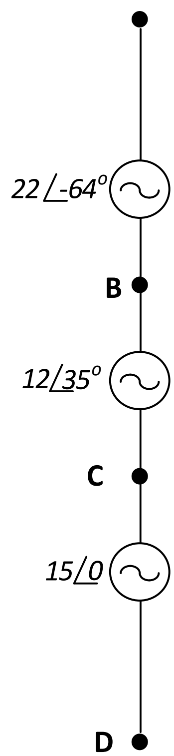
What does the following graph represent? Of course, we see the picture at right and see that it is a graph of torque by cylinder of a V8 engine and the sum of the eight cylinders firing torque curve.



From this, we can see that the sum of various signals (in this case, torque from 8 different cylinders) can add to give an output torque curve for the entire engine. These signals are added together to give the resultant dashed 'total' line.

Some Examples with AC Circuits

The following examples show multiple ac sources added together to form one source. Using rectangular addition, we can find the resultant waveform and magnitude of the output read on the DMM.

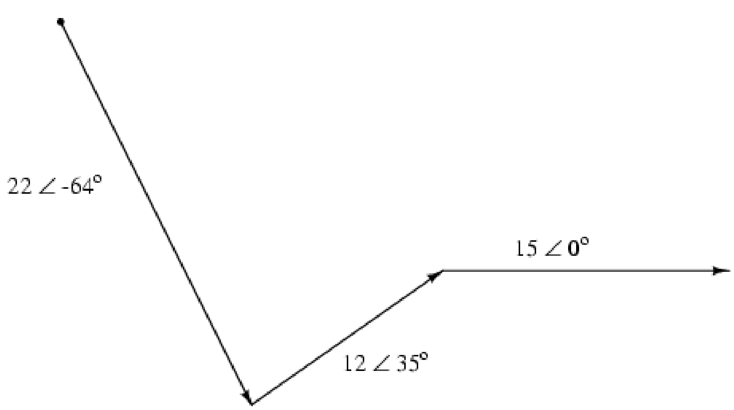


To add the three sources, we need to add the three vectors together. KVL holds:

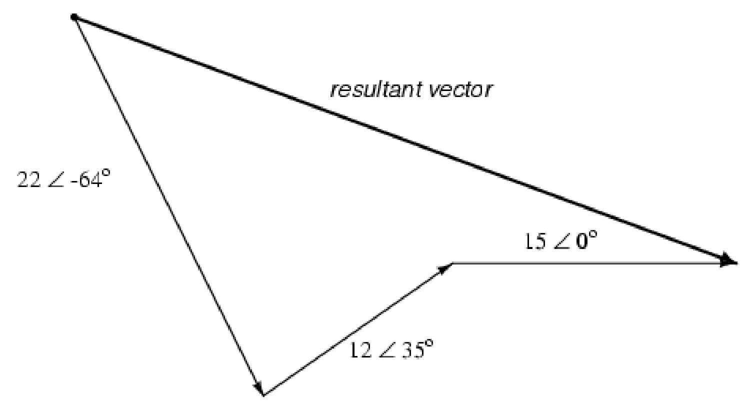
$$V_{total} = V_1 + V_2 + V_3$$

$$V = 22\angle{-64^o} + 12\angle{35^o} + 15\angle{0}$$

These vectors add graphically as follows:



The sum of these vectors is a vector shown below:



We can build the sum by adding the rectangular components of the three vectors:

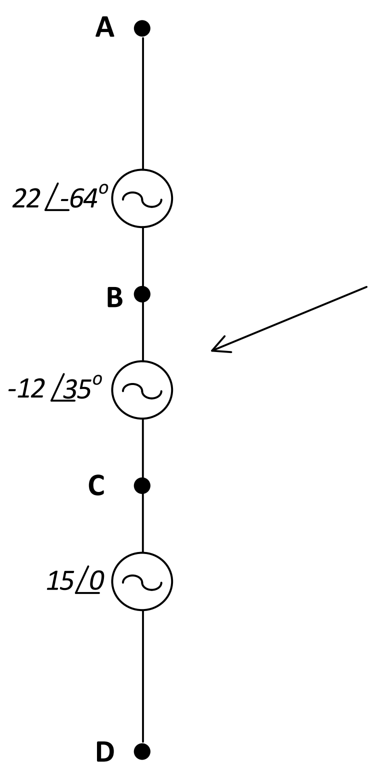
15 /0° = 15 + j0 V

12 /35° = 9.829 + j6.882 V

22 /-64° = 9.644 – j19.773 V

Add the three together:

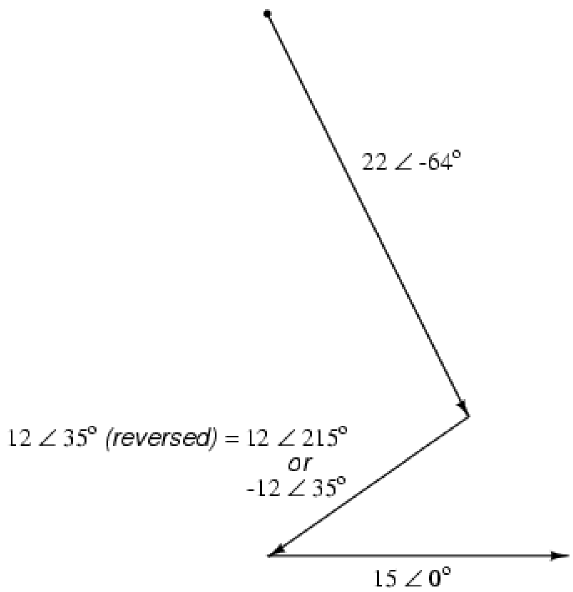
34.474 - j12.890 -> 36.805 /-20.501°



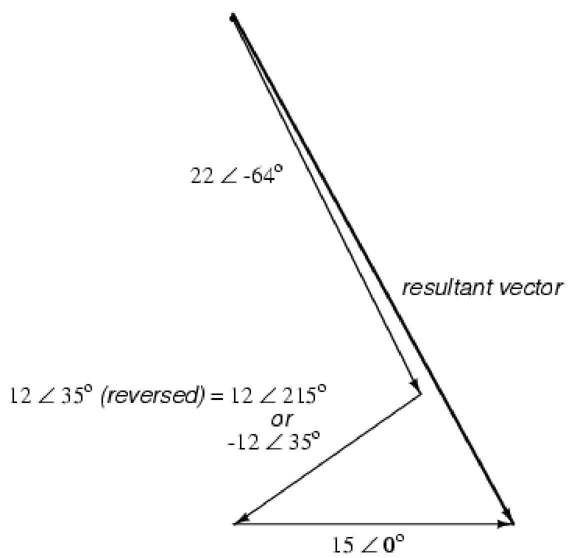
What if we flip the leads on this supply. We change the sign from + to - or change the degrees from 35° to 215°. Either is acceptable.

-12 /35° = *12 /225°*

The three supplies now resemble:



And the resultant vector is:



Resultant Vector

Add the three together:

15 $\underline{/0^{\circ}}$ = 15 + j0 V

112 $\underline{/35^{\circ}}$ = -9.829 - j6.882 V

22 $\underline{/-64^{\circ}}$ = 9.644 - j19.773 V

Summed to equal:

14.8143 – j26.6564 V or 30.4964 V ∠ -60.9368°

Problems regarding Rectangular to Polar Conversion and Vectors

8-1 Convert each of the following rectangular numbers to polar form:

a. 2 + j5
b. -3 + j6
c. -4 – j4
d. 5 – j2

8-2 Convert each of the following rectangular numbers to polar form:

a. 150 + j300
b. -200 + j350
c. -375 – j250
d. 465 – j750

8-3 Convert the following polar numbers to rectangular form

a. $4 \angle 60^\circ$
b. $7 \angle 130^\circ$
c. $5 \angle 250^\circ$
d. $8 \angle 330^\circ$

8-4 Draw the vector in polar and rectangular form for the numbers in 8-1, 8-2 and 8-3.

8-5 Add the second number to the first, then subtract, then multiply, then divide:

a. (2 + j5) + (-3 + J6)
b. (-4 – j4) + (5 – j2)
c. (-3 + j6) + (-4 – j4)
d. (5 – j2) + (3 + j8)

8-6 Add the second number to the first, then subtract, then multiply, then divide:

a. (3 – j2) + (5 + j7)
b. (4 – j5) + (-6 + j8)
c. (-5 + j3) + (3 – j6)
d. (-7 + j2) + (-3 – j9)

8-7 Add, subtract, multiply and divide the second number to the first below:

a. $5 \angle 30^\circ$, $3 \angle 45^\circ$
b. $7 \angle 120^\circ$, $4 \angle 90^\circ$
c. $3 \angle 65^\circ$, $8 \angle -36^\circ$
d. $5 \angle -35^\circ$, $6 \angle -30^\circ$

8-8 For the following, evaluate the expression and find the result in **polar** format:

$$\frac{7+j12}{42-j12}$$

$$\frac{7\angle 35^\circ}{2-j3}$$

Problems using Phasors in Electrical Circuits

8-9 A sine wave has the following equation: $i = 20 \sin (\vartheta + 35^0)$. Find the phasor current.

8-10 Write the sin equation and find the phasor for Fig. 8-1a and b.

8-11 Write the sin equation and draw the sin graph for Fig. 8-1c.

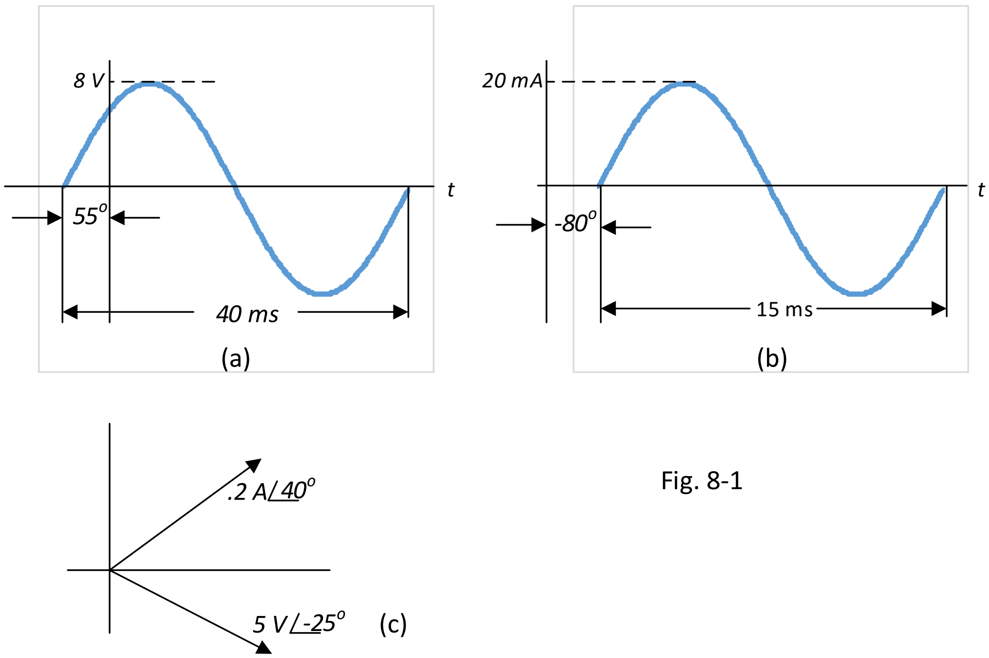
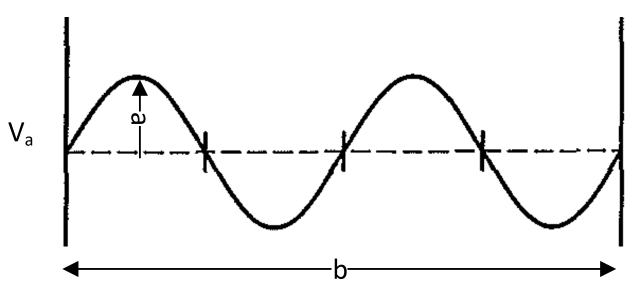
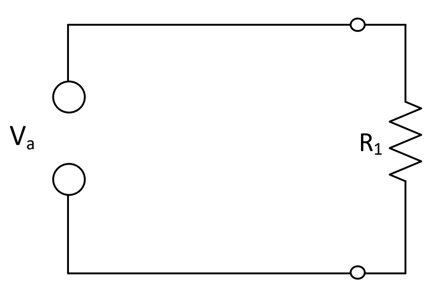


Fig. 8-1

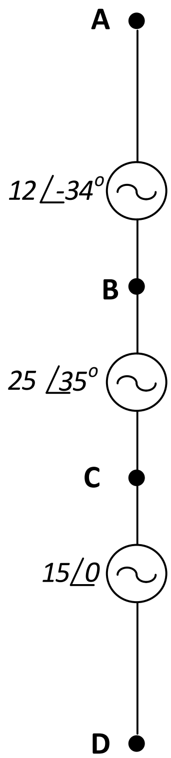
8-12 Use the following values of a and b for V_a to find the sin equation for V_a and then find the power consumed by R_1:

a = 18, b = 48 ms, R_1 = 120 Ω

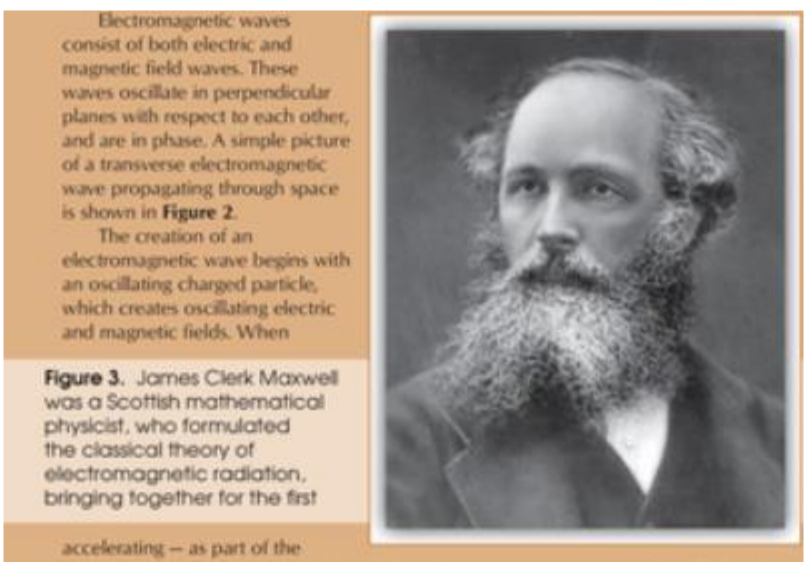




8-13 Use the three voltage sources below to find a single voltage source from A-D. Then invert the 25 V source and find the new total voltage.



8.14 Who are these guys?


Electromagnetic waves consist of both electric and magnetic field waves. These waves oscillate in perpendicular planes with respect to each other, and are in phase. A simple picture of a transverse electromagnetic wave propagating through space is shown in **Figure 2**.

The creation of an electromagnetic wave begins with an oscillating charged particle, which creates oscillating electric and magnetic fields. When

Figure 3. James Clerk Maxwell was a Scottish mathematical physicist, who formulated the classical theory of electromagnetic radiation, bringing together for the first

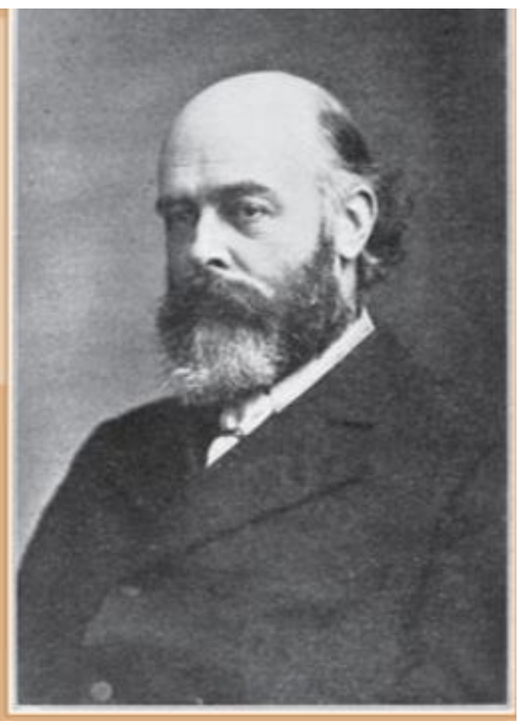
accelerating — as part of the

Figure 5. Oliver Heaviside was a reclusive mathematical genius who spent most of his life on the fringe of the science establishment. In spite of this, he did more than anyone else to shape Maxwell's theory and put Maxwell's equations into their present form.

one that would bring much-needed order to the subject, help newcomers, and establish a solid base for future work.

In 1873, Maxwell published his book "Treatise On Electricity And Magnetism," where he expounded further on many of his ideas. Still in print, it is one of the most renowned books in physics. However, the ***"field"*** concept in the book was alien to

Figure 7. Oliver Joseph Lodge was a British physicist who studied electromagnetic radiation. He made a particularly strong mark in the field of wireless telegraphy. In 1894, he perfected the "coherer:" an electrical device used to detect radio waves. Lodge's version of the coherer greatly improved the detection of radio waves.



capacity, he had the opportunity to learn firsthand about the most advanced and scientifically interesting branch of electrical engineering.

Telegraph equipment of the time used visual cues, so his mild deafness did not play a role. He loved communicating in Morse code, but fixing faults in the cable system was what he really

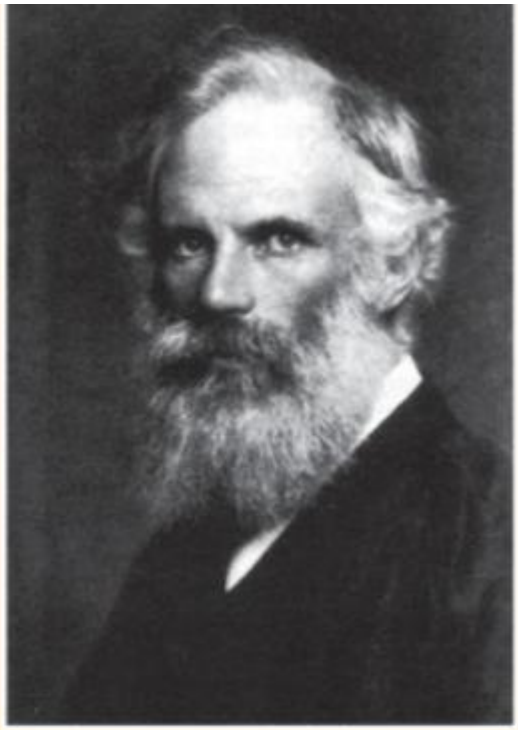


Figure 8. George Francis Fitzgerald was an Irish professor of physics at Trinity College in Dublin, Ireland. He is known for his work in electromagnetic theory and for the Lorentz-Fitzgerald contraction, which became an integral part of Einstein's special theory of relativity.

equations Heaviside's equations. He said that he believed that Maxwell — with good reason — "would have admitted to the necessity of the changes when pointed out to him." Hence, he felt they should be called Maxwell's equations.

Refining a mathematical theory was one thing, but finding experimental evidence to support it was something else. With the help of two other Maxwellians and a little known German researcher, Heinrich Hertz, the fog began to lift.

Up to 1888, Heaviside was doing the same things: reading journals, writing papers that were seldom read, and rarely travelling from his home. One day, he happened upon a report by Oliver Joseph Lodge (**Figure 7**), professor of physics at University College in Liverpool, wherein he was mentioned for his work on Maxwell's theory. Overjoyed to find a kindred spirit, Heaviside promptly wrote to Lodge and found he had another admirer,

was experimenting with
preparation for a talk to

Figure 9. Heinrich Hertz was a brilliant German physicist and experimentalist who demonstrated that the electromagnetic waves predicted by James Clerk Maxwell actually exist. In November 1886, Hertz became the first person to transmit and receive controlled radio waves.

“Permittivity is concerned with electric fields and is the "ability of a material to polarize in response to the (electric) field", while permeability is concerned with magnetic fields and is "the degree of magnetization of a material in response to a magnetic field"

Permittivity of free space: ϵ_0= 8.8541878176E-12 F/m
Permeability of free space: μ_0= 4(pi)E-7 H/m

Interestingly: c=1/ sqrt($\mu_0\epsilon_0$), where c is the speed of light

Let’s start by writing Maxwell’s equations for the vacuum and without charges and currents:

$\nabla\cdot E=0\nabla\cdot E=0$

$\nabla\times E=-\partial B\partial t\nabla\times E=-\partial B\partial t$

$\nabla\cdot B=0\nabla\cdot B=0$

$\nabla\times B=\mu 0\epsilon 0\partial E\partial t\nabla\times B=\mu 0\epsilon 0\partial E\partial t$

Taking the curl of the second equation we have

$\nabla\times(\nabla\times E)=-\nabla\times(\partial B\partial t)\nabla\times(\nabla\times E)=-\nabla\times(\partial B\partial t)$

Using a known identity (curl = grad div $-\nabla^2$), and interchanging second order derivatives, we have

$\nabla(\nabla\cdot E)-\nabla^2E=-\partial\partial t(\nabla\times B).\nabla(\nabla\cdot E)-\nabla^2E=-\partial\partial t(\nabla\times B)$

div E is zero (first equation), and curl B can be replaced from fourth equation:

$\nabla^2E=1/c^2\partial E\partial t^2, \nabla 2E=1/c^2\partial E\partial t^2$

where

$1/c^2=\mu_0\epsilon_0$

This is the wave equation for (transversal) waves propagating at velocity c.

The laws of Electromagnetism predict that should be (electromagnetic) waves moving at speed c. Some of them (those with wavelengths between roughly 0.4 and 0.8μ m) are called light, and therefore c is called the speed of light.

Introduction to Maxwell's Equations

Maxwell's Equations are a set of 4 complicated equations that describe the world of electromagnetics. These equations describe how electric and magnetic fields propagate, interact, and how they are influenced by objects.

James Clerk Maxwell [1831-1879] was an Einstein/Newton-level genius who took a set of known experimental laws (Faraday's Law, Ampere's Law) and unified them into a symmetric coherent set of Equations known as Maxwell's Equations. Maxwell was one of the first to determine the speed of propagation of electromagnetic (EM) waves was the same as the speed of light - and hence to conclude that EM waves and visible light were really the same thing.

Maxwell's Equations are critical in understanding [Antennas] and Electromagnetics. They are formidable to look at - so complicated that most electrical engineers and physicists don't even really know what they mean. Shrouded in complex math (which is likely so "intellectual" people can feel superior in discussing them), true understanding of these equations is hard to come by.

This leads to the reason for this website - an intuitive tutorial of Maxwell's Equations. I will avoid if at all possible the mathematical difficulties that arise, and instead describe what the equations mean. And don't be afraid - the math is so complicated that those who do understand complex vector calculus still cannot apply Maxwell's Equations in anything but the simplest scenarios. For this reason, intuitive knowledge of Maxwell's Equations is far superior than mathematical manipulation-based knowledge. To understand the world, you must understand what equations mean, and not just know mathematical constructs. I believe the accepted methods of teaching electromagnetics and Maxwell's Equations do not produce understanding. And with that, let's say something about these equations.

Maxwell's Equations are laws - just like the law of gravity. These equations are rules the universe uses to govern the behavior of electric and magnetic fields. A flow of electric current will produce a magnetic field. If the current flow varies with time (as in any wave or periodic signal), the magnetic field will also give rise to an electric field. Maxwell's Equations shows that separated charge (positive and negative) gives rise to an electric field - and if this is varying in time as well will give rise to a propagating electric field, further giving rise to a propgating magnetic field.

To understand Maxwell's Equations at a more intuitive level than most Ph.Ds in Engineering or Physics, click through the links and definitions above. You'll find that the complicated math masks an inner elegance to these equations - and you'll learn how the universe operates the Electromagnetic Machine.

1. $\nabla \cdot \mathbf{D} = \rho_V$

2. $\nabla \cdot \mathbf{B} = 0$

3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$

4. $\nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

Here is a tutorial video explaining Maxwell's Equations intuitively:"

Top: Maxwell's Equations

https://www.feynmanlectures.caltech.edu/II_18.html