Complex Numbers and Sinusoidal Steady State

If I needed to describe the distance between Toledo, Ohio and Cincinnati, I would just answer 200 miles. The direction is straight south and I have driven it a number of times. If you, however, would ask the distance to Indianapolis, Indiana, I would have to ask you whether you were driving down I-75 and then turning right at Dayton or if you were driving to Fort Wayne ad then down I-69. You would need to provide both direction and mileage for the description to Indianapolis. I need more information than just a scalar number of miles. I need direction or some other description indicating direction as well as number of miles.

When we deal with scalar numbers, life is simple. However, when we begin to delve into numbers with a direction component, we begin to look at mathematical concepts with more depth. With a circuit containing a battery, we just add the numbers around a loop and solve an equation. With a sinusoidal waveform, the quantities may be simple or complex, depending on the devices in the circuit. We need to learn to deal with complex numbers when dealing with AC circuits if the circuit contains multiple waveforms that may be out of phase with each other or if inductors or capacitors are involved. Whew, I said it! No longer in Kansas, Dorothy!

A complex number is a single mathematical quantity that conveys dimensions of amplitude and phase shift.

Complex numbers are easy to grasp graphically. A line with length and angle can represent a complex number. Several examples are shown in the figure below:

A vector has both magnitude and direction. The reference for these vectors is a Cartesian plane similar to the following ‘vector compass’. Magnitude is measured in vector length and angle is measured counterclockwise from the positive direction.
AC Phase

Below is a picture of sine waves that are out of phase with each other. The A wave leads the B wave by a phase angle that is measurable. We can see this on an oscilloscope. The two waves have the same amplitude and frequency but are out of phase with each other.

If the phase shift were $90^\circ$, we could draw the phasor diagram of the A and B phasors as follows:

If the phase shift were $135^\circ$, we would see a phasor similar to the following:
Various voltage phase shifts can be observed in the following figures:

Various phase shifts are shown below:

Examples of phase shifts.

Remember that the waveforms are at the same frequency. This is implied in all the phasor diagrams. The reference is where you want to place it. There is no absolute reference.
We veer a little off course with the following examples but they may help you to see that phasors are nothing more than a dog and a rabbit running around a circular track with the rabbit always in the lead by a certain amount.

We could say the dogs and rabbit are out of phase. This would be true since the dogs always ‘lag’ the rabbit by a certain amount (in degrees).

You could also see phasors as a cat with the hands moving backward (counter-clockwise). The eyes move, the tail moves and the hands move backwards (only for those who see the minute and hour hands as phasors).

While dogs on a track and cats wagging their tail may seem strange, the idea is to get one to think of phasors as both a stationary vector and a moving arrow with length and angular velocity (just like a dog or a cat).
Rectangular Form of a Complex Number

Vectors have amplitude and magnitude. The following waveforms demonstrate the variable nature of sine waves and their vector representation. Notice that each of the waveforms are assumed to be at zero reference point. Amplitude correlates to relative length of the vector.

Length of Vector represents Signal Amplitude

Simple Vector Addition

Remember that vectors are mathematical objects with magnitude and direction. How would you evaluate the sum of the two voltages below in the figure?

While in-phase sinusoidal waves add the same as dc voltage supplies, if the phase is shifted, the numbers add as complex numbers. Calculate voltage across the sum of two vectors, not a linear sum.
This leads us to the subject of Complex Vector Addition

From the example above, to add the two vectors, we simply add the ‘x’ components of each and the ‘y’ components of each and find the resultant vector. For the $6/0^\circ$ vector, there is only an ‘x’ component of 6. For the $10/60^\circ$ vector, the ‘x’ component is 5.0 and the ‘y’ component is 8.66. Adding, we get the total ‘x’ as 6 + 5 = 11 and for the ‘y’, we get 8.66. Now, find the resultant vector. It is $14/38.21^\circ$.

We use Polar Notation for complex numbers to find the sum of these vectors.

In order to work with complex numbers without drawing vectors, we first need some kind of standard mathematical notation. There are two basic forms of complex number

Polar Form of a Complex Number

We represent sine waves in polar form in order to add them.

We will find that in ac circuits, most of the dc rules still apply, just with complex numbers.

The equation of a sinusoidal wave is:

$$y = M \sin(\varphi + \theta)$$

For current, the equation is:

$$i = I_P \sin(\varphi + \theta)$$

The value at the peak is $I_P$ and the value of phase shift is $\varphi$ (Greek phi).

For voltage, the equation is:

$$v = V_P \sin(\varphi + \theta)$$

The value at the peak is $V_P$ and the value of phase shift is $\varphi$ (Greek phi). The value $\theta$ represents the time-varying signal and can be seen as $2\pi ft$ where $f$ is the frequency and $t$ is time.

The waveform for the current signal is shown in the figure at left while the phasor form is shown at right. The phasor current is the rms value. The relationship between $I_P$ and $I_{RMS}$ is $I_{RMS} = 0.707 I_P$. 

![Waveform and Phasor Diagram](image-url)
With rectangular form, we can more easily add or subtract vectors. We simply add the ‘x’ components and the ‘y’ components and re-combine the result. In general, we try to add and subtract using rectangular form while we multiply and divide using polar form. Some examples of rectangular form of vectors follow:

\[
\text{Length = 5} \\
\text{Angle = 36.87°} \\
\text{Rectangular Format:} \\
4 + j3
\]

\[
\text{Length = 4} \\
\text{Rectangular Format:} \\
-4 + j0
\]

\[
\text{Length = 3} \\
\text{Rectangular Format:} \\
4 + j3
\]

\[
\text{Length = 5} \\
\text{Rectangular Format:} \\
-4 - j3
\]
The ‘x’ component is the real component and the ‘y’ component is the imaginary component. The complex plane is shown below with both real and imaginary axes.

Converting from Polar Form to Rectangular Form

Using a calculator with either R->P or P->R will give the conversion from one form to the other. It is advised for the student to sit down with a calculator and find these functions or a similar method to convert since this will be required for some time in this chapter and the next. Practice...

The above vector may be entered either in rectangular or polar format. It is shown in rectangular format:

\[ 4 + j3 \]

To find the polar equivalent, use a calculator to convert:

Enter: 4, R->P, 3, =
Result: 5

Key in x<>y to get:
36.869

This is written: \( 4 + j3 = 5/36.869 \) (rectangular to polar)
Complex Number Arithmetic

We will be adding complex numbers and it is advised to find a calculator with this capability.

For addition and subtraction of complex numbers, use rectangular form.

\[
\begin{align*}
2 + j5 \\
+ 4 - j3 \\
6 + j2
\end{align*}
\]

For subtraction, use the same method:

\[
\begin{align*}
2 + j5 \\
- (4 - j3) \\
-2 + j8
\end{align*}
\]

For multiplication and division, polar is preferred. For multiplication, multiply magnitudes and add phase angles. For division, divide the magnitudes and subtract the denominator’s phase angle from the numerator’s phase angle.

\[
(35/65^\circ)(10/-10^\circ) = 350/55^\circ
\]

Division of polar-form complex numbers:

\[
\frac{35/65^\circ}{10/-10^\circ} = 3.5/75^\circ
\]

AC “polarity”

If we were to measure the following circuits, we would see the results from the DMM’s display. In other words, if we reference the red lead as plus and measure the plus side of the voltage using this lead, we would read +6.0 V. If the leads were reversed, however, the display would show -6.0 V.
Does the same happen when we test an ac circuit? The answer is that it may if the sources are in
phase. However, per our example above, a phase shift will make our calculation use complex
numbers and the answer will not be a simple addition but rather a vector addition. If we switch the
leads from plus to minus, our answer will be 180° phase shifted which would be what we would
expect. This change, however, cannot be read with a volt meter or simple DMM but only on an
oscilloscope.

What does the following graph represent? Of course, we see the picture at right and see that it is a
graph of torque by cylinder of a V8 engine and the sum of the eight cylinders firing torque curve.

From this, we can see that the sum of various signals (in this case, torque from 8 different cylinders) can
add to give an output torque curve for the entire engine. These signals are added together to give the
resultant dashed ‘total’ line.

Some Examples with AC Circuits

The following examples show multiple ac sources added together to form one source. Using
rectangular addition, we can find the resultant waveform and magnitude of the output read on the
DMM.
To add the three sources, we need to add the three vectors together. KVL holds:

\[ V_{\text{total}} = V_1 + V_2 + V_3 \]

\[ V = 22/\angle -64^\circ + 12/\angle 35^\circ + 15/\angle 0^\circ \]

These vectors add graphically as follows:

The sum of these vectors is a vector shown below:
We can build the sum by adding the rectangular components of the three vectors:

\[
15 \angle 0^\circ = 15 + j0 \text{ V}
\]

\[
12 \angle 35^\circ = 9.829 + j6.882 \text{ V}
\]

\[
22 \angle -64^\circ = 9.644 - j19.773 \text{ V}
\]

Add the three together:

\[
34.474 - j12.890 \rightarrow 36.805 \angle -20.501^\circ
\]

What if we flip the leads on this supply. We change the sign from + to - or change the degrees from 35° to 215°. Either is acceptable.

\[-12 \angle 35^\circ \quad = \quad 12 \angle 225^\circ\]
The three supplies now resemble:

\[12 \angle 35^\circ \text{ (reversed)} = 12 \angle 215^\circ \text{ or } -12 \angle 35^\circ\]

And the resultant vector is:

Add the three together:

\[15 /0^\circ = 15 + j0 \ V\]
\[112 /35^\circ = -9.829 - j6.882 \ V\]
\[22 /-64^\circ = 9.644 - j19.773 \ V\]

Summed to equal:

\[14.8143 - j26.6564 \ V \text{ or } 30.4964 \ V \angle -60.9368^\circ\]
Problems regarding Rectangular to Polar Conversion and Vectors

8-1 Convert each of the following rectangular numbers to polar form:
   a. 2 + j5
   b. -3 + j6
   c. -4 - j4
   d. 5 - j2

8-2 Convert each of the following rectangular numbers to polar form:
   a. 150 + j300
   b. -200 + j350
   c. -375 - j250
   d. 465 - j750

8-3 Convert the following polar numbers to rectangular form
   a. 4 $\angle 60^\circ$
   b. 7 $\angle 130^\circ$
   c. 5 $\angle 250^\circ$
   d. 8 $\angle 330^\circ$

8-4 Draw the vector in polar and rectangular form for the numbers in 8-1, 8-2 and 8-3.

8-5 Add the second number to the first, then subtract, then multiply, then divide:
   a. (2 + j5) + (-3 + 6j)
   b. (-4 - j4) + (5 - j2)
   c. (-3 + j6) + (-4 - j4)
   d. (5 - j2) + (3 + j8)

8-6 Add the second number to the first, then subtract, then multiply, then divide:
   a. (3 - j2) + (5 + j7)
   b. (4 - j5) + (-6 + j8)
   c. (-5 + j3) + (3 - j6)
   d. (-7 + j2) + (-3 - j9)

8-7 Add, subtract, multiply and divide the second number to the first below:
   a. 5 $\angle 30^\circ$, 3 $\angle 45^\circ$
   b. 7 $\angle 120^\circ$, 4 $\angle 90^\circ$
   c. 3 $\angle 65^\circ$, 8 $\angle -36^\circ$
   d. 5 $\angle -35^\circ$, 6 $\angle -30^\circ$

8-8 For the following, evaluate the expression and find the result in polar format:
   \[ \frac{7 + j12}{42 - j12} \]
   \[ \frac{7/\angle35^\circ}{2 - j3} \]
Problems using Phasors in Electrical Circuits

8-9  A sine wave has the following equation: \( i = 20 \sin (\theta + 35^\circ) \). Find the phasor current.

8-10  Write the sin equation and find the phasor for Fig. 8-1a and b.

8-11  Write the sin equation and draw the sin graph for Fig. 8-1c.

8-12  Use the following values of \( a \) and \( b \) for \( V_a \) to find the sin equation for \( V_a \) and then find the power consumed by \( R_1 \):

\[ a = 18, \ b = 48 \text{ ms}, \ R_1 = 120 \ \Omega \]
8.13 Use the three voltage sources below to find a single voltage source from A-D. Then invert the 25 V source and find the new total voltage.

8.14 Who are these guys?
Figure 5. Oliver Heaviside was a reclusive mathematical genius who spent most of his life on the fringe of the science establishment. In spite of this, he did more than anyone else to shape Maxwell’s theory and put Maxwell’s equations into their present form.

one that would bring much-needed order to the subject, help newcomers, and establish a solid base for future work.

In 1873, Maxwell published his book “Treatise On Electricity And Magnetism,” where he expounded further on many of his ideas. Still in print, it is one of the most renowned books in physics. However, the “field” concept in the book was alien to

Figure 7. Oliver Joseph Lodge was a British physicist who studied electromagnetic radiation. He made a particularly strong mark in the field of wireless telegraphy. In 1894, he perfected the “coherer,” an electrical device used to detect radio waves. Lodge’s version of the coherer greatly improved the detection of radio waves.

capacity, he had the opportunity to learn firsthand about the most advanced and scientifically interesting branch of electrical engineering.

Telegraph equipment of the time used visual cues, so his mild deafness did not play a role. He loved communicating in Morse code, but fixing faults in the cable system was what he really
Figure 8. George Francis Fitzgerald was an Irish professor of physics at Trinity College in Dublin, Ireland. He is known for his work in electromagnetic theory and for the Lorentz–Fitzgerald contraction, which became an integral part of Einstein’s special theory of relativity.

Refining a mathematical theory was one thing, but finding experimental evidence to support it was something else. With the help of two other Maxweilians and a little known German researcher, Heinrich Hertz, the fog began to lift.

Up to 1888, Heaviside was doing the same things; reading journals, writing papers that were seldom read, and rarely travelling from his home. One day, he happened upon a report by Oliver Joseph Lodge (Figure 7), professor of physics at University College in Liverpool, wherein he was mentioned for his work on Maxwell’s theory. Overjoyed to find a kindred spirit, Heaviside promptly wrote to Lodge and found he had another admirer.

Figure 9. Heinrich Hertz was a brilliant German physicist and experimentalist who demonstrated that the electromagnetic waves predicted by James Clerk Maxwell actually exist. In November 1886, Hertz became the first person to transmit and receive controlled radio waves.

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