6-24. Show that Eq. 6-121 can be written in the form

$$i = Ke^{-\zeta\omega_n t} \cos(\omega_n\sqrt{1-\zeta^2}\, t + \phi)$$

Give the values for K and ϕ in terms of K_5 and K_6 of Eq. 6-121.

6-25. A switch is closed at $t = 0$ connecting a battery of voltage V with a series RL circuit. (a) Show that the energy in the resistor as a function of time is

$$w_R = \frac{V^2}{R}\left(t + \frac{2L}{R}e^{-Rt/L} - \frac{L}{2R}e^{-2Rt/L} - \frac{3L}{2R}\right) \text{joules}$$

(b) Find an expression for the energy in the magnetic field as a function of time. (c) Sketch w_R and w_L as a function of time. Show the steady-state asymptotes, that is, the values that w_R and w_L approach as $t \to \infty$. (d) Find the total energy supplied by the voltage source in the steady state.

6-26. In the series RLC circuit shown in the accompanying diagram, the frequency of the driving force voltage is

(1) $\omega = \omega_n$ (the undamped natural frequency)

(2) $\omega = \omega_n\sqrt{1-\zeta^2}$ (the natural frequency)

These frequencies are applied in two separate experiments. In each experiment we measure (a) the peak value of the transient current when the switch is closed at $t = 0$, and (b) the maximum value of the steady-state current. (a) In which case (that is, which frequency) is the maximum value of the transient greater? (b) In which case (that is, which frequency) is the maximum value of the steady-state current greater?

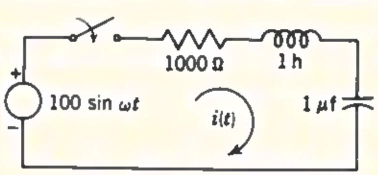

Fig. P6-26

7

The Laplace Transformation

7-1. Introduction

The forerunner of the Laplace transformation method of solving differential equations, the *operational calculus*, was invented by the brilliant English engineer Oliver Heaviside (1850–1925). Heaviside was a practical man and his interest was in the practical solution of electric circuit problems rather than careful justification of his methods. He was gifted with an insight into physical problems that enabled him to pick the correct solution from a number of alternatives. This heuristic point of view drew bitter and perpetual criticism from the leading mathematicians of his time. In the years that followed publication of Heaviside's work, the rigor was supplied by such men as Bromwich, Giorgi, Carson, and others. The basis for substantiating the work of Heaviside was found in the writings of Laplace in 1780. As the years have passed, the structural members of the framework of Heaviside's operational calculus have been replaced, piece by piece, by new members derived by the Laplace transformation. This transformation has provided rigorous substantiation of the operational methods; no important errors have been discovered in Heaviside's results. Indeed, there is a revival of interest in operational methods pioneered by Heaviside, initiated by the work of Mikusiński and sometimes called the *Mikusiński operational calculus*.[1]

[1] Jan Mikusiński, *Operational Calculus* (Pergamon Press, Inc., New York, 1959).

The Laplace transformation method for solving differential equations offers a number of advantages over the classical methods that were discussed in Chapters 4 and 6. For example:

(1) The solution of differential equations is routine and progresses systematically.

(2) The method gives the total solution—the particular integral and the complementary function—in one operation.

(3) Initial conditions are automatically specified in the transformed equations. Further, the initial conditions are incorporated into the problem as one of the first steps rather than as the last step.

What is a transformation? The *logarithm* is an example of a transformation that we have used in the past. Logarithms greatly simplify such operations as multiplication, division, extracting roots, and raising quantities to powers. Suppose that we have two numbers, given to seven-place accuracy, and we are required to find the product, maintaining the accuracy of the given numbers. Rather than just multiplying the two numbers together, we transform these numbers by taking their logarithms. These logarithms are added (or subtracted in the case of division). The resulting sum itself has little meaning. However, if we perform an *inverse transformation*, if we find the antilogarithm, then we have the desired numerical result. The direct division looks more straightforward, but our experience has been that the use of the logarithm often saves time. If the simple problem of multiplying two numbers is not convincing, consider evaluating $(1437)^{0.1328}$ without logarithms!

A flow sheet of the operation of using logarithms to find a product or a quotient is shown in Fig. 7-1. The individual steps are: (1) find the logarithm of each separate number, (2) add or subtract the numbers to obtain the sum of logarithms, and (3) take the antilogarithm to obtain the product or quotient. This is roundabout compared with *direct* multiplication or division, yet we use logarithms to advantage, particularly when a good table of logarithms is available.

The flow sheet idea may be used to illustrate what we will do in using the Laplace transformation to solve a differential equation. The flow sheet for the Laplace transformation is shown in Fig. 7-1(*b*) with a block corresponding to every block of the logarithm flow sheet considered above. The steps will be as follows. (1) Start with an integrodifferential equation and find the corresponding Laplace *transform*. This is a mathematical process, but there are tables of transforms just as there are tables of logarithms (and one is included in this chapter). (2) The transform is manipulated algebraically after the initial conditions are inserted. The result is a *revised transform*. As step (3), we perform an inverse Laplace transformation to give us the solution.

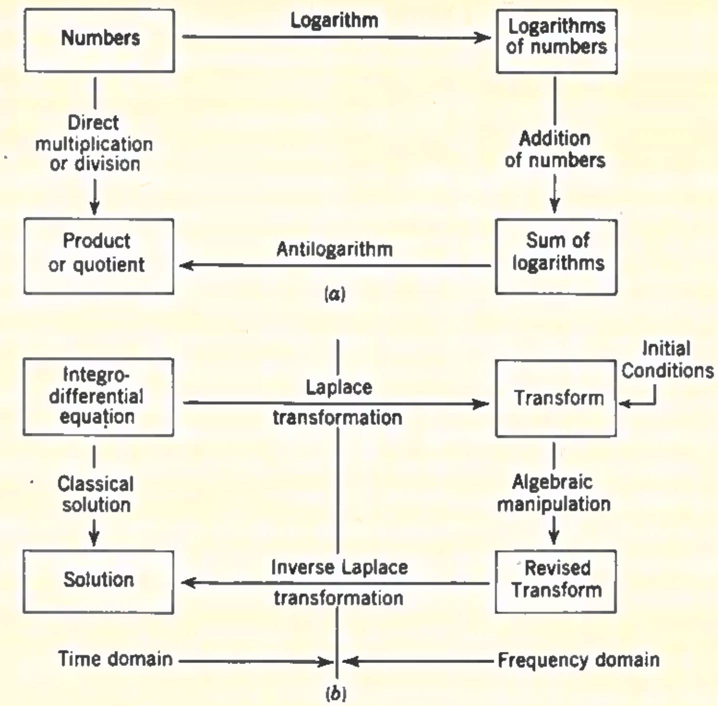

Fig. 7-1. Comparison of logarithms and the Laplace transformation.

In this step, we also can use a table of transforms, just as we use the table of logarithms in the corresponding step for logarithms. The flow sheet reminds us that *there is another way:* the classical solution. It looks more direct (and sometimes it is for simple problems). For complicated problems, an advantage will be found for the Laplace transformation, just as an advantage was found for the use of logarithms.

7-2. The Laplace Transformation

To construct a Laplace transform for a given function of time $f(t)$, we first multiply $f(t)$ by e^{-st}, where s is a complex number, $s = \sigma + j\omega$. This product is integrated with respect to time from zero to infinity. The result is the Laplace transform of $f(t)$, which is designated $F(s)$. Denoting the Laplace transformation by the script letter $\mathcal{L}$ (in order to reserve L for inductance), the Laplace transformation is given by the expression

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad (7\text{-}1)$$

The letter $\mathcal{L}$ can be replaced by the words "the Laplace transform of" in the above expression. In order that $f(t)$ be transformable, it is sufficient that

$$\int_0^\infty |f(t)|e^{-\sigma_1 t}\,dt < \infty \tag{7-2}$$

for a real, positive σ_1. Although Eq. 7-1 may be a ra her formidable appearing integral at first glance, the actual evaluation of $F(s)$ for a given $f(t)$ is usually not difficult. Furthermore, once the transform of a function is found, it need not be found again for each new problem, but can be tabulated for future use. The time function $f(t)$ and its transform $F(s)$ are called a *transform pair*. Various workers have compiled extensive tables of transform pairs, so that Eq. 7-1 will seldom be used in solving problems in practice.

Two parts of the last paragraph require further examination. (1) How serious is the limitation imposed by Eq. 7-2 that must be satisfied for $f(t)$ to have a transform? (2) For a given $F(s)$, how do we find the corresponding $f(t)$: Can transform pairs be used in reverse?

The restriction of Eq. 7-2 is satisfied by most $f(t)$ encountered in engineering, since $e^{-\sigma t}$ is a "powerful reducing agent" as a multiplier of $f(t)$. Thus we may show by l'Hospital's rule that

$$\lim_{t\to\infty} t^n e^{-\sigma t} = 0, \qquad \sigma > 0 \tag{7-3}$$

such that the integral of the product, for $n = 1$, is

$$\int_0^\infty te^{-\sigma t}\,dt = \frac{1}{\sigma^2}, \qquad \sigma > 0 \tag{7-4}$$

and the integral for other values of n similarly remains finite for $\sigma \neq 0$.

An example of a function that does not satisfy Eq. 7-2 is e^{at^2} or, in general, e^{at^n}; there is no value of σ for which the integral of Eq. 7-2 for these $f(t)$ remain finite. Now such functions are seldom required to describe the driving function in engineering problems. Furthermore, a generator would produce this function for only a limited range of values of t, and thereafter would saturate at a constant value. The function we have just described which is

$$\begin{aligned} v &= e^{at^2}, & 0 \le t \le t_0 \\ &= K, & t > t_0 \end{aligned} \tag{7-5}$$

does, of course, satisfy Eq. 7-2.

Example 1

As an example of the evaluation of Eq. 7-1, consider the *unit step function* introduced by Heaviside. This function is described by the equation

$$\begin{aligned} u(t) &= 1, & t \ge 0 \\ &= 0, & t < 0 \end{aligned} \tag{7-6}$$

as shown in Fig. 7-2. Such a notation is convenient for representing the closing of a switch at $t = 0$; if a battery of voltage V_0 is connected to a network at $t = 0$, then the driving voltage may be represented as $V_0u(t)$ without the necessity of mentioning the presence of a switch (or showing it in the schematic diagram). For $V_0 = 1$, we have

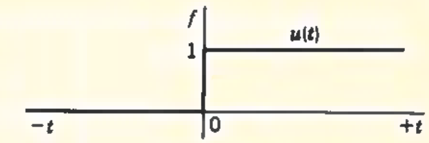

Fig. 7-2. The unit step function, $f = u(t)$.

$$\mathcal{L}[u(t)] = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \tag{7-7}$$

Similarly,

$$\mathcal{L}[V_0u(t)] = \frac{V_0}{s} \tag{7-8}$$

Example 2

As a second example of the calculation of a transform, let $f(t) = e^{at}$, where a is a constant. Substituting into Eq. 7-1, we have

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}, \qquad \sigma_1 > a \tag{7-9}$$

Thus e^{at} and $1/(s-a)$ constitute a transform pair.

These two computations form the beginning of a table of transform pairs as shown below.

TABLE OF TRANSFORM PAIRS

$f(t)$	$F(s)$
$u(t)$	$1/s$
e^{at}	$\frac{1}{s-a}$

We may now turn to our second question which concerns finding $f(t)$ from $F(s)$. The inverse Laplace transformation is given by the *complex inversion integral*,

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_1-j\infty}^{\sigma_1+j\infty} F(s)e^{st}\,ds \tag{7-10}$$

which is a contour integral where the path of integration, known as the Bromwich path, is along the vertical line $s = \sigma_1$ from $-j\infty$ to $j\infty$, as shown in Fig. 7-3. On the figure is also shown the *abscissa of convergence*, which is the number designated as a in Eq. 7-9. For the proper evaluation of the

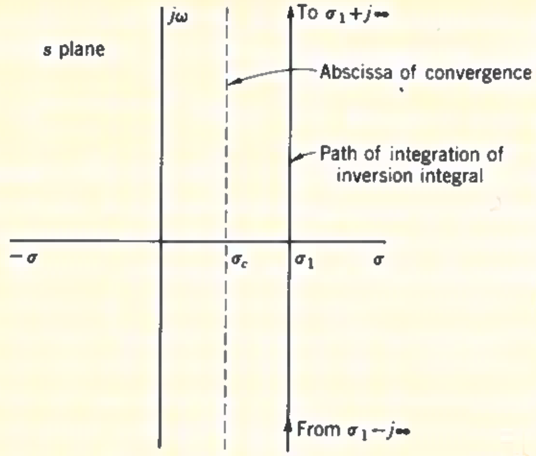

Fig. 7-3. Figure showing the abscissa of convergence which relates to the inverse Laplace transformation.

inversion integral, we require that $\sigma_1 > \sigma_c$ or that the path of integration in Eq. 7-10 be to the right of the abscissa of convergence. In terms of our using the Laplace transformation, what do these results mean? In terms of Eq. 7-9, the fact that a σ_1 may be chosen greater than a implies that the transform $F(s)$ exists since Eq. 7-2 is satisfied. If we use the inversion integral to compute $f(t)$ by Eq. 7-10, then a proper value of σ_1 could be chosen by knowing σ_c. However, another property of the Laplace transformation makes it unnecessary for us to use the inversion integral in most cases. This property is the *uniqueness* of the Laplace transformation:[2] there cannot be two different functions having the same Laplace transformation, $F(s)$. That being the case, we may use the table of transform pairs to find $f(t)$, provided we can find the necessary form of $F(s)$ in the table. We use the symbol $\mathcal{L}^{-1}$ to indicate the *inverse Laplace transformation*. Then

$$\mathcal{L}^{-1}\{\mathcal{L}[f(t)]\} = \mathcal{L}^{-1}[F(s)] = f(t) \tag{7-11}$$

7-3. Some Basic Theorems for the Laplace Transformation

(*1*) *Transforms of Linear Combinations.* If $f_1(t)$ and $f_2(t)$ are two functions of time and a and b are constants, then

$$\mathcal{L}[af_1(t) + bf_2(t)] = aF_1(s) + bF_2(s) \tag{7-12}$$

This theorem is established with Eq. 7-1. It follows from the fact that the

[2] See S. Seshu and N. Balabanian, *Linear Network Analysis* (John Wiley & Sons, Inc., New York, 1959), p. 553.

integral of a sum of terms is equal to the sum of the integrals of the terms; that is,

$$\begin{aligned}\mathcal{L}[af_1(t) + bf_2(t)] &= \int_0^\infty [af_1(t) + bf_2(t)]e^{-st}\,dt \\ &= a\int_0^\infty f_1(t)e^{-st}\,dt + b\int_0^\infty f_2(t)e^{-st}\,dt \\ &= aF_1(s) + bF_2(s)\end{aligned} \tag{7-13}$$

We will make use of this theorem in finding the Laplace transformation of the sum of terms that appear in network equations.

Example 3

As an example of the use of this result, let us find the transform of $\cos \omega t$ and $\sin \omega t$. From Euler's identity, Eq. 6-37, which is

$$e^{\pm j\omega t} = \cos \omega t \pm j \sin \omega t \tag{7-14}$$

we see that by adding the two equations of Eqs. 7-14, we have

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \tag{7-15}$$

and by subtracting,

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \tag{7-16}$$

Since the transform of the exponential function is

$$\mathcal{L}[e^{\pm j\omega t}] = \frac{1}{s \mp j\omega}, \qquad \sigma_1 > 0 \tag{7-17}$$

we see that

$$\mathcal{L}[\cos \omega t] = \frac{1}{2}\left(\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right) = \frac{s}{s^2 + \omega^2}, \qquad \sigma_1 > 0 \tag{7-18}$$

and

$$\mathcal{L}[\sin \omega t] = \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \frac{\omega}{s^2 + \omega^2}, \qquad \sigma_1 > 0 \tag{7-19}$$

These results may be added to our collection of transform pairs.

(*2*) *Transforms of Derivatives.* From the defining equation for the Laplace transformation, we write

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = \int_0^\infty \frac{d}{dt}f(t)e^{-st}\,dt \tag{7-20}$$

This equation may be integrated by parts by letting

$$u = e^{-st} \quad \text{and} \quad dv = df(t) \tag{7-21}$$

in the equation

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \qquad (7\text{-}22)$$

Then

$$du = -se^{-st}\,dt \quad \text{and} \quad v = f(t) \qquad (7\text{-}23)$$

so that the transform of a derivative becomes

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = sF(s) - f(0+) \qquad (7\text{-}24)$$

provided $\lim_{t\to\infty} f(t)e^{-st} = 0$, which follows from l'Hospital's rule, provided that $f(t)$ and its successive derivatives are finite at $t = \infty$ and $\sigma > 0$.

To find the transform of the second derivative, we follow a similar procedure but make use of the result of Eq. 7-24. Since

$$\frac{d^2}{dt^2}f(t) = \frac{d}{dt}\frac{d}{dt}f(t) \qquad (7\text{-}25)$$

then

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s\mathcal{L}\left[\frac{df}{dt}(t)\right] - \frac{df}{dt}(0+)$$

$$= s[sF(s) - f(0+)] - \frac{df}{dt}(0+) \qquad (7\text{-}26)$$

$$= s^2F(s) - sf(0+) - \frac{df}{dt}(0+)$$

In this expression, the quantity $df/dt(0+)$ is the derivative of $f(t)$ evaluated at $t = 0+$ (the time immediately after switching action is initiated). The general expression for an nth derivative is

$$\mathcal{L}\frac{d^nf(t)}{dt^n} = s^nF(s) - s^{n-1}f(0+) - s^{n-2}\frac{df}{dt}(0+) - \dots - \frac{d^{n-1}}{dt^{n-1}}f(0+) \qquad (7\text{-}27)$$

(3) *Transforms of Integrals.* The transform for an integral is found by starting from the definition

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \int_0^\infty \left[\int_0^t f(t)\,dt\right]e^{-st}\,dt \qquad (7\text{-}28)$$

The integration is carried out by parts where we let

$$u = \int_0^t f(t)\,dt, \qquad du = f(t)\,dt \qquad (7\text{-}29)$$

and

$$dv = e^{-st}\,dt, \qquad v = -\frac{1}{s}e^{-st} \qquad (7\text{-}30)$$

Hence

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = -\frac{e^{-st}}{s}\int_0^t f(t)\,dt\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt \qquad (7\text{-}31)$$

Now the first term vanishes since e^{-st} approaches zero for infinite t, and at the lower limit

$$\int_0^t f(t)\,dt\Big|_{t=0} = 0 \qquad (7\text{-}32)$$

Hence we conclude that

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \qquad (7\text{-}33)$$

Now the formulation of the Kirchhoff laws for a network often involve an integral with limits from $-\infty$ to t. Such integrals may be divided into two parts

$$\int_{-\infty}^t f(t)\,dt = \int_{-\infty}^0 f(t)\,dt + \int_0^t f(t)\,dt \qquad (7\text{-}34)$$

where the first term on the right of this equation is a constant. When $f(t)$ is current, then this integral is the initial value of charge, $q(0+)$, and when $f(t)$ is voltage, then the integral is flux linkages $\psi(0+) = Li(0+)$. In either case, this term should be included in the equation formulation; the transform of a constant, $q(0+)$ is, from Eq. 7-8,

$$\mathcal{L}[q(0+)] = \frac{q(0+)}{s} \qquad (7\text{-}35)$$

and a similar equation may be written for $\psi(0+)$.

7-4. Examples of the Solution of Problems with the Laplace Transformation

With the short table of transforms that has been given on page 163 and the three basic theorems that have been derived in the previous section, we are now equipped to solve a network problem (elementary as yet, to be sure) using the Laplace transformation.

Example 4

For this example, we will write the Kirchhoff voltage law for a series RC network shown in Fig. 7-4. It will be assumed that the switch K is closed at $t = 0$. This information will be included in the formation of the network equations by writing the voltage expression as $Vu(t)$. Hence

$$\frac{1}{C}\int_{-\infty}^t i\,dt + Ri = Vu(t) \qquad (7\text{-}36)$$

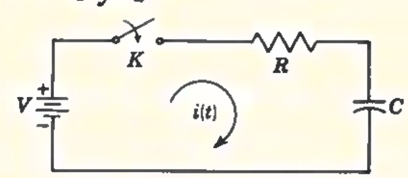

Fig. 7-4. *RC* series network for Example 4.

This is the integral equation we wish to solve. The transforms of the linear combination of terms is

$$\frac{1}{C}\left[\frac{I(s)}{s} + \frac{q(0+)}{s}\right] + RI(s) = V \cdot \frac{1}{s} \qquad (7\text{-}37)$$

In terms of the flow chart of Fig. 7-1, we have found the Laplace transformation of the integral equation and there has resulted a transform expression. The required initial conditions are automatically specified and may be inserted as the second step (rather than as the final step as in differential equations solved by classical methods). Now $q(0+)$ is the charge on the capacitor at $t = 0+$. If the capacitor is initially uncharged, $q(0+) = 0$ and the last equation reduces to the form

$$I(s)\left(\frac{1}{Cs} + R\right) = \frac{V}{s} \qquad (7\text{-}38)$$

The next step, again according to the flow chart, is algebraic manipulation. The objective of this manipulation is to solve for $I(s)$. This is accomplished by multiplying by s and dividing by R to give

$$I(s) = \frac{V/R}{s + 1/RC} \qquad (7\text{-}39)$$

which is a "revised transform" expression. The next step on our flow chart is to perform the inverse Laplace transformation and obtain the solution. That is,

$$\mathcal{L}^{-1}[I(s)] = \mathcal{L}^{-1}\left(\frac{V/R}{s + 1/RC}\right) = i(t) \qquad (7\text{-}40)$$

Using the second transform pair of our short table, the solution is

$$i(t) = \frac{V}{R}e^{-t/RC} \qquad (7\text{-}41)$$

This is the complete solution. The arbitrary constant emerges evaluated (and has the magnitude V/R).

Example 5

As our second example, consider the *RL* series circuit shown in Fig. 7-5

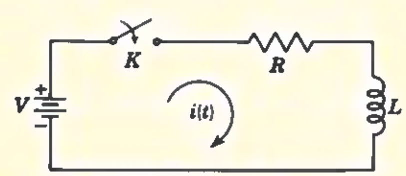

Fig. 7-5. ***RL*** **series network for Example 5.**

with the switch closed at $t = 0$. The differential equation for the circuit is, by Kirchhoff's law,

$$L\frac{di}{dt} + Ri = Vu(t) \qquad (7\text{-}42)$$

The corresponding transform equation is

$$L[sI(s) - i(0+)] + RI(s) = \frac{V}{s} \qquad (7\text{-}43)$$

The initial condition specified by the last equation is $i(0+)$, the current after the switch is closed. Because of the inductance, $i(0+) = 0$. Our equation may now be manipulated to solve for $I(s)$; thus

$$I(s) = \frac{V}{L}\frac{1}{s(s + R/L)} \qquad (7\text{-}44)$$

This transform, however, is not in our short table. We need something new (or a larger table). Notice that this term is made up of the product of the term $(1/s)$ and the term $[1/(s + R/L)]$. We know the inverse Laplace transformation of each of these individual terms. This suggests that the inverse operation could be performed if there were some way to break the transform terms into several parts. As an attempt to perform this operation, let us try the following expansion:

$$\frac{V/L}{s(s + R/L)} = \frac{K_0}{s} + \frac{K_1}{s + R/L} \qquad (7\text{-}45)$$

In this equation K_0 and K_1 are unknown coefficients. As the first step, let us simplify the equation by putting all terms over a common denominator. Then

$$\frac{V}{L} = K_0\left(s + \frac{R}{L}\right) + K_1 s \qquad (7\text{-}46)$$

By equating coefficients of like functions, we obtain a set of linear algebraic equations:

$$K_0 \cdot \frac{R}{L} = \frac{V}{L}, \qquad K_0 + K_1 = 0 \qquad (7\text{-}47)$$

From these two equations, we find the required values for K_0 and K_1:

$$K_0 = \frac{V}{R} \quad \text{and} \quad K_1 = -\frac{V}{R} \qquad (7\text{-}48)$$

This algebraic manipulation has permitted Eq. 7-44 to be written

$$I(s) = \frac{V}{L}\frac{1}{s(s + R/L)} = \frac{V}{R}\left[\frac{1}{s} - \frac{1}{s + R/L}\right] \qquad (7\text{-}49)$$

We have transform pairs corresponding to each of these expressions. The current as a function of time is found by taking the inverse Laplace transformation of the individual expressions; thus

$$i(t) = \frac{V}{R}\left(\mathcal{L}^{-1}\frac{1}{s} - \mathcal{L}^{-1}\frac{1}{s + R/L}\right) \tag{7-50}$$

or

$$i(t) = \frac{V}{R}(1 - e^{-Rt/L}) \tag{7-51}$$

This is the final (time-domain) solution. The method we used to expand a transform into the sum of several separate parts is known under the heading of *partial fraction expansion*. It is this subject that we study next.

7-5. Partial Fraction Expansion

The examples in the last section have suggested the general procedure in applying the Laplace transformation to the solution of integrodifferential equations. A differential equation of the general form

$$a_0\frac{d^ni}{dt^n} + a_1\frac{d^{n-1}i}{dt^{n-1}} + \dots + a_{n-1}\frac{di}{dt} + a_ni = v(t) \tag{7-52}$$

becomes, as a result of the Laplace transformation, an algebraic equation which may be solved for the unknown as

$$I(s) = \frac{\mathcal{L}[v(t)] + \text{initial condition terms}}{a_0s^n + a_1s^{n-1} + \dots + a_{n-1}s + a_n} \tag{7-53}$$

The general form of this equation is a quotient of polynomials in s. Let the numerator and denominator polynomials be designated $P(s)$ and $Q(s)$, respectively, as

$$I(s) = \frac{P(s)}{Q(s)} \tag{7-54}$$

Note that $Q(s) = 0$ is the characteristic equation of Chapter 6. If the transform term $P(s)/Q(s)$ can now be found in a table of transform pairs, the solution $i(t)$ can be written directly. In general, however, the transform expression for $I(s)$ must be broken into simpler terms before any practical transform table can be used.

As the first step in the expansion of the quotient $P(s)/Q(s)$, we check to see that the order[3] of the polynomial P is less than that of Q. If this condition is not fulfilled, divide the numerator by the denominator to obtain an expansion in the form

$$\frac{P(s)}{Q(s)} = B_0 + B_1s + B_2s^2 + \dots + B_{m-n}s^{m-n} + \frac{P_1(s)}{Q(s)} \tag{7-55}$$

where m is the order of the numerator and n the order of the denominator.

[3] Beginning with Chapter 9, we will use *degree* rather than *order*. See footnote 3 of Chapter 6.

The new function $P_1(s)/Q(s)$ has now been "prepared" and the order-rule is satisfied.

Example 6

Consider the quotient

$$\frac{P(s)}{Q(s)} = \frac{s^2 + 2s + 2}{s + 1} \tag{7-56}$$

By direct division,

$$\begin{array}{r} s+1)\ \ s^2 + 2s + 2 \quad (s + 1 \\ \underline{s^2 + s\phantom{{}+2}} \\ s + 2 \\ \underline{s + 1} \\ 1 \end{array}$$

or

$$\frac{s^2 + 2s + 2}{s + 1} = 1 + s + \frac{1}{s + 1} \tag{7-57}$$

so that in Eq. 7-55, $B_0 = 1$, $B_1 = 1$, and $P_1(s)/Q(s) = 1/(s + 1)$.

Next, we factor the denominator polynomial, $Q(s)$,

$$Q(s) = a_0s^n + a_1s^{n-1} + \dots + a_n = a_0(s - s_1)\dots(s - s_n) \tag{7-58}$$

or, very compactly,

$$Q(s) = a_0\prod_{j=1}^{n}(s - s_j) \tag{7-59}$$

where Π indicates a product of factors, and $s_1, s_2, \dots, s_n$ are the n roots of the equation $Q(s) = 0$. Now the possible form of these roots was discussed in Chapter 6: (1) real and simple (or distinct) roots, (2) conjugate complex roots, and (3) multiple roots. We will consider these possibilities separately.

(1) If all roots of $Q(s) = 0$ are simple, then the partial fraction expansion is

$$\frac{P_1(s)}{(s - s_1)(s - s_2)\dots(s - s_n)} = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} + \dots + \frac{K_n}{s - s_n} \tag{7-60}$$

where the K's are real constants called *residues*.

(2) If a root of $Q(s) = 0$ is of multiplicity r, the partial fraction expansion for the repeated root is

$$\frac{P_1(s)}{(s - s_1)^r} = \frac{K_{11}}{s - s_1} + \frac{K_{12}}{(s - s_1)^2} + \dots + \frac{K_{1r}}{(s - s_1)^r} \tag{7-61}$$

and there will be similar terms for every other repeated root.

(3) An important special rule may be given for two roots which form a complex conjugate pair. For this case, the partial fraction expansion is

$$\frac{P_1(s)}{Q_1(s)(s+\alpha+j\omega)(s+\alpha-j\omega)}$$
$$= \frac{K_1}{(s+\alpha+j\omega)} + \frac{K_1^*}{(s+\alpha-j\omega)} + \cdots \quad (7\text{-}62)$$

where K_1^* is the complex conjugate of K_1. In other words, when the roots are conjugates, so are the partial fraction expansion coefficients. An expansion of the type shown above is necessary for each pair of complex conjugate roots.

In an expansion of a quotient of polynomials by partial fractions, it may be necessary to use a combination of the three rules given above. Several examples will illustrate the expansion and the determination of the K's.

Example 7

Consider the quotient of polynomials,

$$I(s) = \frac{2s+3}{s^2+3s+2} \quad (7\text{-}63)$$

The first step is to factor the denominator polynomial and then expand by the appropriate rule. For this example, the expansion is

$$\frac{2s+3}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)} \quad (7\text{-}64)$$

since the roots are real and simple. As the first step, multiply the equation by $(s+1)$ as

$$\frac{(2s+3)(s+1)}{(s+1)(s+2)} = K_1\frac{s+1}{s+1} + K_2\frac{s+1}{s+2} \quad (7\text{-}65)$$

or, canceling common factors,

$$\frac{2s+3}{s+2} = K_1 + K_2\frac{s+1}{s+2} \quad (7\text{-}66)$$

In this equation, the coefficient K_1 is not multiplied by any function of s. Now s is merely an algebraic factor that can have any value. If $s = -1$, the coefficient of K_2 reduces to zero and we can solve for K_1 as

$$K_1 = \left.\frac{2s+3}{s+2}\right|_{s=-1} = \frac{-2+3}{-1+2} = 1 \quad (7\text{-}67)$$

To evaluate K_2 and to follow the same pattern, multiply Eq. 7-64 by $(s+2)$ to obtain

$$\frac{2s+3}{s+1} = K_1\frac{s+2}{s+1} + K_2 \quad (7\text{-}68)$$

To evaluate K_2, we set $s = -2$ in order to reduce the coefficient of K_1 to zero. Then

$$K_2 = \left.\frac{2s+3}{s+1}\right|_{s=-2} = \frac{-4+3}{-2+1} = 1 \quad (7\text{-}69)$$

The result of the partial fraction expansion is thus

$$\frac{2s+3}{s^2+3s+2} = \frac{1}{s+1} + \frac{1}{s+2} \quad (7\text{-}70)$$

The expansion should always be checked by combining the two terms.

Example 8

For this example, consider a quotient of polynomials with repeated denominator roots:

$$\frac{s+2}{(s+1)^2} = \frac{K_{11}}{s+1} + \frac{K_{12}}{(s+1)^2} \quad (7\text{-}71)$$

Multiplying by $(s+1)^2$ gives

$$s+2 = (s+1)K_{11} + K_{12} \quad (7\text{-}72)$$

and when $s = -1$, K_{12} is readily evaluated as $K_{12} = 1$. If we attempt to follow the same pattern to evaluate K_{11}, trouble develops. That is,

$$\frac{s+2}{s+1} = K_{11} + \frac{K_{12}}{s+1} \quad (7\text{-}73)$$

If, in this equation, $s = -1$, one term becomes infinite and K_{11} cannot be evaluated. However, the problem can be resolved if we return to Eq. 7-72 and differentiate with respect to s:

$$1 + 0 = K_{11} + 0 \quad \text{or} \quad K_{11} = 1$$

The constants are now evaluated and the partial fraction expansion is

$$\frac{s+2}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2} \quad (7\text{-}74)$$

Again, this expansion can be checked, in this case by multiplying the first term in the expansion by $(s+1)/(s+1)$.

Example 9

This example will illustrate the expansion of a quotient of polynomials where the denominator roots are a complex conjugate pair. Consider the quotient

$$\frac{1}{s^2+2s+5} = \frac{K_1}{(s+1-j2)} + \frac{K_1^*}{(s+1+j2)} \quad (7\text{-}75)$$

Multiplying the equation by $s+1-j2$ and then letting $s = -1+j2$ gives $K_1 = -j\frac{1}{4}$; similarly $K_1^* = j\frac{1}{4}$, and the expansion is

$$\frac{1}{s^2+2s+5} = \frac{-j\frac{1}{4}}{(s+1-j2)} + \frac{j\frac{1}{4}}{(s-1+j2)} \quad (7\text{-}76)$$

To use some transform tables, such terms should be revised by completing the square. In this example,

$$(s^2 + 2s + 5) = (s^2 + 2s + 1) + 4 = (s + 1)^2 + 2^2$$

so that

$$\frac{1}{s^2 + 2s + 5} = \frac{1}{(s + 1)^2 + 2^2} \tag{7-77}$$

In the general form $[(s + a)^2 + b^2]$, a is the real part of the root, and b is the imaginary part.

7-6. Heaviside's Expansion Theorem

The method of partial fraction expansion illustrated by the last three examples is known as the Heaviside partial fraction expansion method. To generalize the method, again consider the case in which $Q(s)$ has only distinct roots. Let

$$\frac{P_1(s)}{Q(s)} = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} + \frac{K_3}{s - s_3} + \ldots + \frac{K_n}{s - s_n} \tag{7-78}$$

Then any of the coefficients $K_1, K_2, K_3, \ldots, K_n$ can be evaluated by multiplying by the denominator of that coefficient and setting s to the value of the root of the denominator. In other words, to find the coefficient K_j,

$$K_j = \left[(s - s_j)\frac{P_1(s)}{Q(s)}\right]_{s=s_j} \tag{7-79}$$

To consider a general case of r-repeated roots, let

$$\frac{P(s)}{Q(s)} = \frac{R(s)}{(s - s_j)^r} = \frac{K_{j1}}{s - s_j} + \frac{K_{j2}}{(s - s_j)^2} + \cdots \frac{K_{jn}}{(s - s_j)^n} + \ldots + \frac{K_{jr}}{(s - s_j)^r} \tag{7-80}$$

where n is any term in the partial fraction expansion and $R(s)$ is defined as

$$R(s) = \frac{P(s)}{Q(s)}(s - s_j)^r \tag{7-81}$$

Multiplying Eq. 7-80 by $(s - s_j)^r$ gives

$$R(s) = K_{j1}(s - s_j)^{r-1} + K_{j2}(s - s_j)^{r-2} + \ldots + K_{jr} \tag{7-82}$$

From this equation, we can visualize the method to be used to evaluate each coefficient. If we let $s = s_j$, all terms in the equation disappear except K_{jr}, which can be evaluated. Next, differentiate the equation once with respect to s. The term K_{jr} will vanish, but $K_{j,r-1}$ will remain without a multiplying function of s. Again, $K_{j,r-1}$ can be evaluated by letting $s = s_j$. To find the general term K_{jn}, differentiate Eq. 7-82 $(r - n)$ times and let $s = s_j$; then

$$K_{jn} = \frac{1}{(r - n)!}\frac{d^{r-n}R(s)}{ds^{r-n}}\bigg|_{s=s_j} \tag{7-83}$$

or

$$K_{jn} = \frac{1}{(r - n)!}\frac{d^{r-n}}{ds^{r-n}}\left\{\frac{P(s)}{Q(s)}(s - s_j)^r\right\}\bigg|_{s=s_j} \tag{7-84}$$

Example 10

The actual use of this idea is easier than might appear from the complexity of this general equation. For example, consider

$$\frac{2s^2 + 3s + 2}{(s + 1)^3} = \frac{K_{11}}{(s + 1)} + \frac{K_{12}}{(s + 1)^2} + \frac{K_{13}}{(s + 1)^3} \tag{7-85}$$

Multiplying the equation by $(s + 1)^3$, we have

$$2s^2 + 3s + 2 = K_{11}(s + 1)^2 + K_{12}(s + 1) + K_{13} \tag{7-86}$$

From this equation,

$$K_{13} = 2s^2 + 3s + 2\bigg|_{s=-1} = 2 - 3 + 2 = 1 \tag{7-87}$$

Next, we differentiate with respect to s to obtain

$$4s + 3 = 2K_{11}(s + 1) + K_{12} \tag{7-88}$$

so that

$$K_{12} = 4s + 3\bigg|_{s=-1} = -1 \tag{7-89}$$

Again, we differentiate the last equation to give

$$4 = 2K_{11} \quad \text{or} \quad K_{11} = 2 \tag{7-90}$$

The partial fraction expansion is

$$\frac{2s^2 + 3s + 2}{(s + 1)^3} = \frac{2}{s + 1} + \frac{-1}{(s + 1)^2} + \frac{1}{(s + 1)^3} \tag{7-91}$$

Example 11

If $Q(s)$ contains both simple and repeated roots, a combination of both rules may be used. As an example, let

$$\frac{P(s)}{Q(s)} = \frac{s + 2}{(s + 1)^2(s + 3)} \tag{7-92}$$

The form of the partial fraction expansion is

$$\frac{s + 2}{(s + 1)^2(s + 3)} = \frac{K_{11}}{s + 1} + \frac{K_{12}}{(s + 1)^2} + \frac{K_2}{s + 3} \tag{7-93}$$

In this expansion, K_2 may be evaluated by Eq. 7-79 and K_{11} and K_{12} may be found from Eq. 7-84; then

$$K_2 = \frac{s + 2}{(s + 1)^2}\bigg|_{s=-3} = -\frac{1}{4} \tag{7-94}$$

Multiplying Eq. 7-93 by $(s + 1)^2$, we have

$$\frac{s+2}{s+3} = K_{11}(s+1) + K_{12} + \frac{(s+1)^2}{s+3}K_2 \tag{7-95}$$

The constant K_{12} is evaluated directly by letting $s = -1$; thus

$$K_{12} = \left.\frac{s+2}{s+3}\right|_{s=-1} = \frac{1}{2} \tag{7-96}$$

and K_{11} will be found by differentiating Eq. 7-95 before letting $s = -1$:

$$\frac{(s+3)\cdot 1 - (s+2)\cdot 1}{(s+3)^2} = K_{11} + K_2 \frac{d}{ds}\left[\frac{(s+1)^2}{s+3}\right] \tag{7-97}$$

The coefficient of K_2 vanishes when $s = -1$ because an $(s+1)$ term remains common to all terms in the numerator. In the example

$$\frac{d}{ds}\left[\frac{(s+1)^2}{s+3}\right] = \frac{(s+3)2(s+1) - (s+1)^2\cdot 1}{(s+3)^2} \tag{7-98}$$

and this term vanishes when $s = -1$, because each term in the differentiation contains $(s+1)$. This is always the case, since the order of the multiplying factor $(s - s_j)^r$ is higher than the number of times differentiation is required.

By using these methods, all the coefficients of the partial fraction expansion can be found and the transform equation can be written

$$F(s) = \sum_{j=1}^{n} \frac{K_j}{s - s_j} \tag{7-99}$$

for simple roots of $Q(s) = 0$ and as

$$F(s) = \sum_{k=1}^{r} \frac{K_{jk}}{(s - s_j)^k} \tag{7-100}$$

for a single root, s_j, repeated r times. The corresponding $f(t)$ may now be found, for the general case, by taking the inverse Laplace transformation of $F(s)$ as

$$f(t) = \mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \sum_{j=1}^{n} (s - s_j)\frac{P(s)}{Q(s)}e^{st}\Big|_{s=s_j} \tag{7-101}$$

as the time-domain solution for simple roots. Likewise, for repeated roots,

$$f(t) = e^{s_j t} \sum_{n=1}^{r} \frac{1}{(r-n)!} \frac{d^{r-n}R(s_j)}{ds^{r-n}} \frac{t^{n-1}}{(n-1)!} \tag{7-102}$$

where s_j, in this equation, is *the root* that is repeated r times. By using both equations for the case of both simple and repeated roots, a general solution is obtained in the form originally given as *Heaviside's expansion theorem.*

The method of the Heaviside partial fraction expansion may be used to give a simplified procedure for finding the inverse transform of the terms for a conjugate complex pair of roots. Suppose that these roots have a real part α and imaginary parts $\pm\,\omega$. The first coefficient is evaluated by the procedure,

$$K_1 = \frac{P(s)}{Q(s)}(s + \alpha - j\omega)\Big|_{s=-\alpha+j\omega} = Re^{j\theta} \tag{7-103}$$

and the second as

$$K_1^* = \frac{P(s)}{Q(s)}(s + \alpha + j\omega)\Big|_{s=-\alpha-j\omega} = Re^{-j\theta} \tag{7-104}$$

The inverse transformation of these two terms gives

$$f_1(t) = Re^{j\theta}e^{(-\alpha+j\omega)t} + Re^{-j\theta}e^{(-\alpha-j\omega)t} \tag{7-105}$$

This equation may be rearranged to the form

$$f_1(t) = 2Re^{-\alpha t}\left[\frac{e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}}{2}\right] \tag{7-106}$$

$$= 2Re^{-\alpha t}\cos(\omega t + \theta)$$

The factors R and θ are the magnitude and the phase angle of K_1 in Eq. 7-103. K_1 is the residue associated with the root with a positive imaginary part.

7-7. Examples of Solution by the Laplace Transformation

Example 12

As an example of the total solution, now that the methods of partial fraction expansion have been reviewed, consider the differential equation

$$\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 5i = 5u(t) \tag{7-107}$$

The Laplace transformation of this differential equation is

$$\left[s^2I(s) - si(0+) - \frac{di}{dt}(0+)\right] + 4[sI(s) - i(0+)] + 5I(s) = \frac{5}{s} \tag{7-108}$$

Notice that the required initial conditions are automatically specified in this equation. We must know, from the physical system, $i(0+)$ and $di/dt(0+)$. Suppose the following values are found:

$$i(0+) = 1 \quad \text{and} \quad \frac{di}{dt}(0+) = 2 \tag{7-109}$$

Inserting these initial conditions simplifies the transform equation to

$$I(s)(s^2 + 4s + 5) = \frac{5}{s} + s + 6 \tag{7-110}$$

or

$$I(s) = \frac{s^2 + 6s + 5}{s(s^2 + 4s + 5)} \tag{7-111}$$

This equation may be expanded by partial fractions as

$$I(s) = \frac{s^2 + 6s + 5}{s(s + 2 + j1)(s + 2 - j1)} = \frac{K_1}{s} + \frac{K_2}{s + 2 - j1} + \frac{K_2^*}{s + 2 + j1} \tag{7-112}$$

To evaluate K_1, multiply the equation by s and let $s = 0$. Then

$$K_1 = \left.\frac{s^2+6s+5}{s^2+4s+5}\right|_{s=0} = 1 \tag{7-113}$$

To evaluate K_2, multiply the equation by $(s+2-j1)$ and let $s = -2+j1$ as

$$K_2 = \left.\frac{s^2+6s+5}{s(s+2+j1)}\right|_{s=-2+j1}$$
$$= \frac{-4+j2}{(-2+j1)(j2)} = \frac{2}{j2} = -j = e^{-j90°} \tag{7-114}$$

The complete partial fraction expansion becomes

$$I(s) = \frac{1}{s} + \frac{-j}{s+2-j1} + \frac{j}{s+2+j1} \tag{7-115}$$

To obtain $i(t)$ from this transform equation, we take the inverse Laplace transformation of the first term and use Eq. 7-106 with $R = 1$ and $\theta = -90°$ for the second and third terms to give the solution

$$i(t) = 1 + 2e^{-2t}\sin t, \qquad t \geq 0 \tag{7-116}$$

Example 13

For this example, consider a series *RLC* circuit with the capacitor initially charged to voltage V_0 as indicated in Fig. 7-6. The differential equation for the current $i(t)$ is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0 \tag{7-117}$$

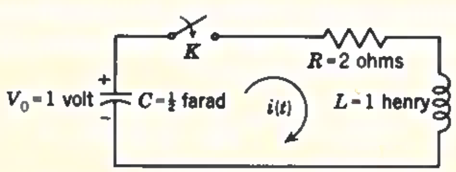

Fig. 7-6. *RLC* series network for Example 13.

and the corresponding transform equation is

$$L[sI(s) - i(0+)] + RI(s) + \frac{1}{Cs}[I(s) + q(0+)] = 0 \tag{7-118}$$

The parameters have been specified as $C = \frac{1}{2}$ farad, $R = 2$ ohms, and $L = 1$ henry. The initial current $i(0+) = 0$ because of the inductor, and if C is initially charged to voltage V_0 (with the given polarity),

$$\frac{q(0+)}{Cs} = -\frac{V_0}{s} \tag{7-119}$$

or $-1/s$ if $V_0 = 1$ volt. The transform equation for $I(s)$ then becomes

$$I(s) = \frac{1}{s^2+2s+2} \tag{7-120}$$

or, completing the square,

$$I(s) = \frac{1}{(s+1)^2+1} \tag{7-121}$$

Using transform pair 15 of page 180,

$$i(t) = \mathcal{L}^{-1}I(s) = e^{-t}\sin t \cdot u(t) \tag{7-122}$$

Example 14

In the network shown in Fig. 7-7, the switch is closed at $t = 0$. With the network parameter values shown, the Kirchhoff voltage equations are

$$\frac{di_1}{dt} + 20i_1 - 10i_2 = 100u(t), \tag{7-123}$$

$$\frac{di_2}{dt} + 20i_2 - 10i_1 = 0 \tag{7-124}$$

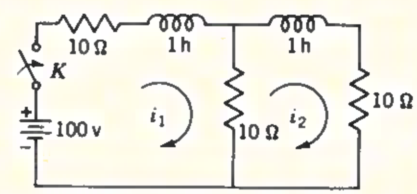

Fig. 7-7. Network for Example 14. The switch is closed at $t = 0$ with zero inductor current at $t = 0-$.

If the network is unenergized before the switch is closed, both i_1 and i_2 are initially zero, and the transform equations may be written

$$(s+20)I_1(s) - 10I_2(s) = \frac{100}{s}, \qquad -10I_1(s) + (s+20)I_2(s) = 0 \tag{7-125}$$

Suppose that we are required to find the current i_2 as a function of time. The transform current $I_2(s)$ may be found from the last two algebraic equations by determinants as

$$I_2(s) = \frac{\begin{vmatrix} s+20 & 100/s \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{1000}{s(s^2+40s+300)} \tag{7-126}$$

The partial fraction expansion of this equation is

$$\frac{1000}{s(s+10)(s+30)} = \frac{3.33}{s} - \frac{5}{s+10} + \frac{1.67}{s+30} \tag{7-127}$$

Table 7-1.

Table of Transforms

$f(t)$*	$F(s)$
1. $u(t)$ 1	$\frac{1}{s}$
2. t	$\frac{1}{s^2}$
3. $\frac{t^{n-1}}{(n-1)!}$ n = integer	$\frac{1}{s^n}$
4. e^{at}	$\frac{1}{s-a}$
5. te^{at}	$\frac{1}{(s-a)^2}$
6. $\frac{1}{(n-1)!}t^{n-1}e^{at}$	$\frac{1}{(s-a)^n}$
7. $\frac{1}{a-b}(e^{at}-e^{bt})$	$\frac{1}{(s-a)(s-b)}$
8. $\frac{e^{-at}}{(b-a)(c-a)} + \frac{e^{-bt}}{(a-b)(c-b)} + \frac{e^{-ct}}{(a-c)(b-c)}$	$\frac{1}{(s+a)(s+b)(s+c)}$
9. $1 - e^{+at}$	$\frac{-a}{s(s-a)}$
10. $\frac{1}{\omega}\sin \omega t$	$\frac{1}{s^2+\omega^2}$
11. $\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12. $1 - \cos \omega t$	$\frac{\omega^2}{s(s^2+\omega^2)}$
13. $\sin(\omega t + \theta)$	$\frac{s \sin\theta + \omega\cos\theta}{s^2+\omega^2}$
14. $\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
15. $e^{-\alpha t}\sin \omega t$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
16. $e^{-\alpha t}\cos\omega t$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
17. $\sinh \alpha t$	$\frac{\alpha}{s^2-\alpha^2}$
18. $\cosh \alpha t$	$\frac{s}{s^2-\alpha^2}$

* All $f(t)$ should be thought of as being multiplied by $u(t)$, *i.e.*, $f(t) = 0$ for $t < 0$.

The inverse Laplace transformation gives $i_2(t)$ as

$$i_2(t) = 3.33 - 5e^{-10t} + 1.67e^{-30t}, \qquad t \geq 0 \qquad (7\text{-}128)$$

which is the required solution.

The properties of Laplace transforms derived in this chapter are summarized in the next chapter in Table 8-1 (together with additional properties derived in Chapter 8). A short table of Laplace transforms is given as Table 7-1; a more extensive table is given in Appendix D.

FURTHER READING

The Laplace transformation is discussed in many engineering textbooks including the following: Franklin F. Kuo, *Network Analysis and Synthesis* (John Wiley & Sons, New York, 1962), Chapters 5 and 6; David K. Cheng, *Analysis of Linear Systems* (Addison-Wesley Publishing Co., Reading, Mass., 1959), Chapters 6 and 7; B. J. Ley, S. G. Lutz, and C. F. Rehberg, *Linear Circuit Analysis* (McGraw-Hill, Inc., New York, 1959), Chapter 8; Edward Peskin, *Transient and Steady-State Analysis of Electric Networks* (D. Van Nostrand Co., Inc., Princeton, N.J., 1961), Chapters 2 and 3; and Roger Legros and A. V. J. Martin, *Transform Calculus for Electrical Engineers* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1961). An excellent treatment of the subject is given by C. R. Wylie, Jr., *Advanced Engineering Mathematics* (*2nd Ed.*) (McGraw-Hill, Inc., New York, 1960), Chapter 8.

For interesting reading in the historical aspects of the subject, the reader should consult the summary titled "The Work of Oliver Heaviside" by Behrend in an appendix to E. J. Berg's *Heaviside's Operational Calculus* (*2nd Ed.*) (McGraw-Hill, Inc., New York, 1936), pp. 173–208. Heaviside's original writings have been reprinted as *Electromagnetic Theory* (Dover Publications, Inc., New York, 1950) and contain an extensive presentation of the method. See also the historical notes in Appendix C of M. F. Gardner and J. L. Barnes, *Transients in Linear Systems* (John Wiley & Sons, Inc., New York, 1942).

PROBLEMS

7-1. Verify that the Laplace transform of $\cos \omega t$ determined from Eq. 7-1 is

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2+\omega^2}$$

In each of the problems that follow, repeat the procedure of Prob. 7-1 for the various transform pairs of the following table.

	$f(t)$ for $t > 0$	$F(s)$
7-2.	t^2	$\frac{2}{s^3}$

7-3.	$\sinh \alpha t$	$\dfrac{\alpha}{s^2 - \alpha^2}$
7-4.	$\cosh \alpha t$	$\dfrac{s}{s^2 - \alpha^2}$
7-5.	$e^{-\alpha t} \sin \omega t$	$\dfrac{\omega}{(s+\alpha)^2 + \omega^2}$
7-6.	$e^{-\alpha t} \cos \omega t$	$\dfrac{s+\alpha}{(s+\alpha)^2 + \omega^2}$
7-7.	$\sin (\omega t + \theta)$	$\dfrac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2}$
7-8.	$\cos (\omega t + \theta)$	$\dfrac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$

For the following $f(t)$, determine the other member of the transform pair, $F(s) = \mathcal{L}[f(t)] = P(s)/Q(s)$ where $P(s)$ and $Q(s)$ are polynomials in s.

7-9. $f_1(t) = \sin^2 t$
7-10. $f_2(t) = t \cos at$
7-11. $f_3(t) = [1/(2a)] \sin at$
7-12. $f_4(t) = \cos^2 t$
7-13. $f_5(t) = [1/(2a^3)](\sinh at - \sin at)$
7-14. $f_6(t) = [1/(2a^2)] \sin at \sinh at$
7-15. $f_7(t) = [1/(2a^2)](\cosh at - \cos at)$
7-16. $f_8(t) = [1/(2a)](\sin at + at \cos at)$

7-17. In the network shown in the figure, C is charged to V_0, and the switch K is closed at $t = 0$. Solve for the current $i(t)$ using the Laplace transformation method.

7-18. In the network shown in the figure, the switch K is moved from position a to position b at $t = 0$, a steady state having previously been established at position a. Solve for the current $i(t)$, using the Laplace transformation method.

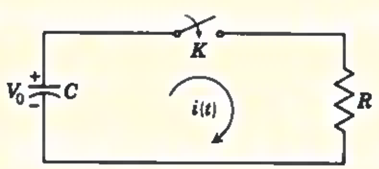

Fig. P7-17

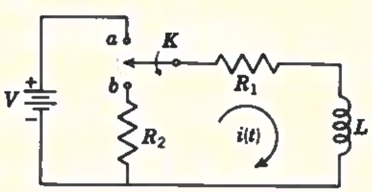

Fig. P7-18

7-19. In the network shown, C is initially charged to V_0. The switch K is closed at $t = 0$. Solve for the current $i(t)$, using the Laplace transformation method.

7-20. In the network shown, the switch K is moved from position a to position b at $t = 0$ (a steady state existing in position a before $t = 0$). Solve for the current $i(t)$, using the Laplace transformation method.

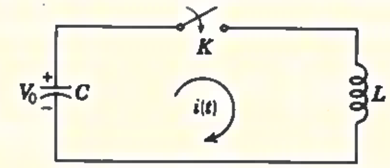

Fig. P7-19

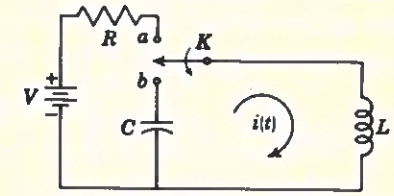

Fig. P7-20

7-21. Work Prob. 4-2 using the Laplace transformation method.

7-22. Work Prob. 4-3 using the Laplace transformation method of this chapter.

7-23. Work Prob. 4-4 using the Laplace transformation method.

7-24. Work Prob. 4-5 using the Laplace transformation method rather than the classical method of Chapter 4.

7-25. Check the following equations by completing a partial fraction expansion. Determine which two expansions in the set are in error.

(a) $\dfrac{2s}{s^2 - 1} = \dfrac{1}{s+1} + \dfrac{1}{s-1}$

(b) $\dfrac{7s + 2}{s^3 + 3s^2 + 2s} = \dfrac{1}{s} + \dfrac{2}{s+2} + \dfrac{-3}{s+1}$

(c) $\dfrac{5s + 13}{s^2 + 5s + 6} = \dfrac{2}{s+3} + \dfrac{3}{s+2}$

(d) $\dfrac{s^2}{s-1} = \dfrac{1}{s-1} + s + 1$

(e) $\dfrac{2(s+1)}{s^2+1} = \dfrac{1 + j1}{s + j1} + \dfrac{1 - j1}{s - j1}$

(f) $\dfrac{s^2 + 4s + 1}{s(s+1)^2} = \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2} + \dfrac{2}{s}$

(g) $\dfrac{3s^3 - s^2 - 3s + 2}{s^2(s-1)^2} = \dfrac{1}{s} + \dfrac{2}{s^2} + \dfrac{2}{s-1} + \dfrac{1}{(s-1)^2}$

(h) $\dfrac{s^3 - 5s^2 + 9s + 9}{s^2(s^2+9)} = \dfrac{1}{s} + \dfrac{1}{s^2} + \dfrac{-j}{s + j3} + \dfrac{+j}{s - j3}$

7-26. Expand the following functions as partial fractions:

(a) $F_1(s) = \dfrac{(s+1)(s+3)}{s(s+2)(s+4)}$

(b) $F_2(s) = \dfrac{(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}$

(c) $F_3(s) = \dfrac{s}{s^2(s+1)^2(s+2)}$

(d) $F_4(s) = \dfrac{1}{s^3+2s^2+2s+1}$

(e) $F_5(s) = \dfrac{s+4}{(s+1)^2(s+3)^2}$

(f) $F_6(s) = \dfrac{s^2(s+3)}{(s+1)(s+2)^4}$

7-27. Verify the following inverse Laplace transformations $\mathcal{L}^{-1}F(s) = f(t)$:

(a) $\mathcal{L}^{-1}\dfrac{3s}{(s^2+1)(s^2+4)} = \cos t - \cos 2t$

(b) $\mathcal{L}^{-1}\dfrac{s+1}{s^2+2s} = \frac{1}{2}(1+e^{-2t})$

(c) $\mathcal{L}^{-1}\dfrac{1}{s(s^2-2s+5)} = \frac{1}{5}[1+\frac{1}{2}e^{t}(-2\cos 2t + \sin 2t)]$

(d) $\mathcal{L}^{-1}\dfrac{1}{(s+1)(s+2)^2} = e^{-t} - e^{-2t}(1+t)$

(e) $\mathcal{L}^{-1}\dfrac{1}{s^2(s^2-1)} = -1 - t^2/2 + \cosh t$

(f) $\mathcal{L}^{-1}\dfrac{s^2+2s+1}{(s+2)(s^2+4)} = \frac{1}{8}e^{-2t} + \frac{7}{8}\cos 2t + \frac{1}{8}\sin 2t$

(g) $\mathcal{L}^{-1}\dfrac{s^2}{(s^2+1)^2} = \frac{1}{2}t\cos t + \frac{1}{2}\sin t$

7-28. Solve the differential equations given in Prob. 6-4 using the Laplace transformation method.

7-29. Solve the differential equations of Prob. 6-5 using the Laplace transformation method.

Solve the following equations using the Laplace transformation method:

7-30. $\dfrac{d^2i}{dt^2} - i = 25 + e^{2t}$

7-31. $\dfrac{d^2i}{dt^2} + 4i = \sin t - \cos 2t$

7-32. $\dfrac{d^2i}{dt^2} + \dfrac{di}{dt} = t^2 + 2t, \qquad i(0+) = 4, \qquad \dfrac{di}{dt}(0+) = -2$

7-33. Solve the differential equation given in Prob. 6-16 using the Laplace transformation method.

7-34. In the series RLC circuit shown, the applied voltage is $v(t) = \sin t$ for $t > 0$. For the element values specified, find $i(t)$ if the switch K is closed at $t = 0$.

7-35. At $t = 0$, a switch is closed, connecting a voltage source $v = V\sin \omega t$ to a series RL circuit. By the method of the Laplace transformation, show that the current is given by the equation

$$i = \frac{V}{Z}\sin(\omega t - \phi) + \frac{\omega L V}{Z^2}e^{-Rt/L}$$

where

$$Z = \sqrt{R^2 + (\omega L)^2} \quad \text{and} \quad \phi = \tan^{-1}\frac{\omega L}{R}$$

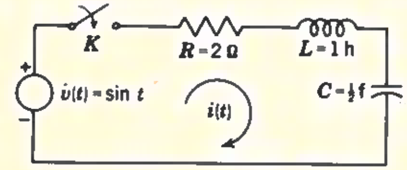

Fig. P7-34

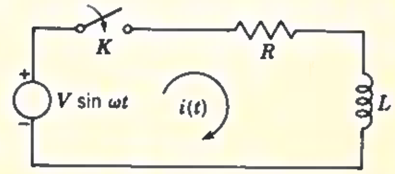

Fig. P7-35

7-36. Dr. L. A. Woodbury of the University of Utah School of Medicine has made use of an electrical analog in studies of convulsions. In the network shown in the figure, the following quantities are duals: C_1 represents the volume of drug-containing fluid, R_1 is the "resistance" to the passage of the drug from the pool to the blood stream, C_2 represents the volume of the blood stream, and R_2 is equivalent to the body's excretion mechanism (kidney, etc.). The concentration of the drug dose is represented as V_0 and the voltage $v_a(t)$ at node a is analogous to the amount of drug in the blood stream. The analog network has the advantage that the elements may be readily changed and the effects studied (to say nothing of the saving of cats). Find the transform equation for $V_a(s)$ with the coefficient of the highest-order term normalized to unity.

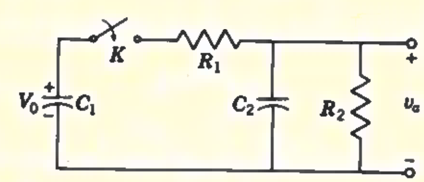

Fig. P7-36

7-37. This problem is a continuation of Prob. 7-36 concerning Dr. Woodbury's analog. The following constants for the network are selected: $C_1 = 1\ \mu f$, $C_2 = 8\ \mu f$, $R_1 = 9$ megohms, and $R_2 = 5$ megohms. If $V_0 = 100$

volts and the switch is closed at $t = 0$, solve for $v_A(t)$, the equivalent of the concentration of drug in the blood stream, as a function of time.

7-38. Find the time t_m when the concentration of drug in the blood stream for Prob. 7-37 is a maximum. (This information is desired so that a second dose may be given at that time to build up the concentration to the point where a convulsion is induced.)

7-39. If a second dose (the voltage equivalent having a magnitude of 100 volts) is injected at $t = t_m$ as found in Prob. 7-38, what will be v_A as a function of time, and what will be the maximum v_A obtained? (*Note:* In giving the second dose we will assume that the total voltage is then 100 volts plus the voltage on the plates at the time the addition is made.)

7-40. In the network shown, the switch K is closed at $t = 0$ with the network previously unenergized. For the element values shown on the diagram: (a) find $i_1(t)$, (b) find $i_2(t)$.

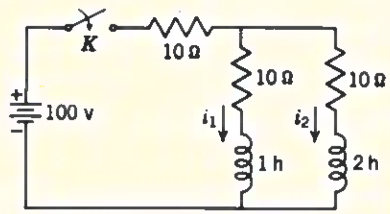

Fig. P7-40

7-41. With switch K in a position a, the network shown in the figure attains equilibrium. At time $t = 0$, the switch is moved to position b. Find the voltage across R_2 as a function of time.

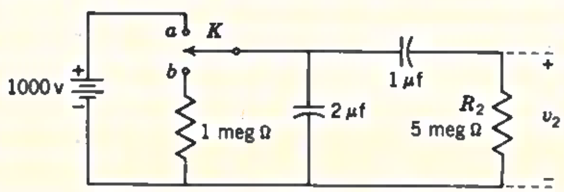

Fig. P7-41

7-42. (a) Find $i_1(t)$ resulting from closing the switch at $t = 0$ with the circuit previously unenergized. The circuit constants are: $L_1 = 1$ henry, $L_2 = 4$ henrys, $M = 2$ henrys, $R_1 = R_2 = 1$ ohm, $V = 1$ volt. (b) Repeat part (a) in solving for $i_2(t)$.

7-43. In the network given in the figure, the current source is described by $i_1 = 10^{-3}e^{-t}u(t)$ amp. For the element values given, determine $v_2(t)$,

assuming that all elements are initially unenergized. Sketch $v_2(t)$ using an enlarged time scale for small values of t.

7-44. The network given in the figure contains a current-controlled voltage source. For the element values given and with $v_1(t) = 5u(t)$, determine $v_a(t)$ if the network is not energized at $t = 0$. Let $K_1 = -3$.

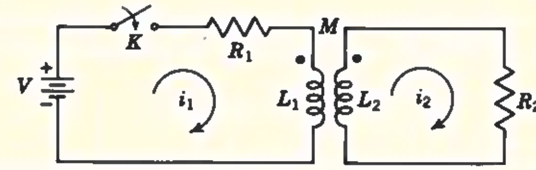

Fig. P7-42

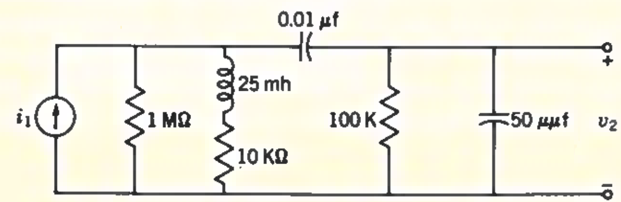

Fig. P7-43

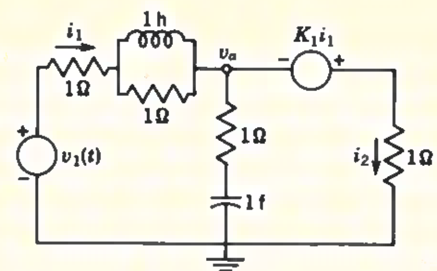

Fig. P7-44

7-45. Using the conditions given for the network of Prob. 7-44, solve for $i_2(t)$ if $K_1 = +3$.

7-46. Show that

$$\mathcal{L}\frac{f(t)}{t} = \int_s^\infty F(s)\,ds$$

7-47. Find

$$\mathcal{L}^{-1}\frac{n!}{s(s+1)(s+2)\dots(s+n)}$$

by partial fraction expansion, and show that the answer may be given in the closed form $(1 - e^{-t})^n$.